A representation formula for the best constant in the Sobolev immersion $W^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega)$

Grey Ercole *

Departamento de Matemática - ICEx, Universidade Federal de Minas Gerais,
Av. Antônio Carlos 6627, Caixa Postal 702, 30161-970, Belo Horizonte, MG, Brazil

April 18, 2013

Abstract

Let $\lambda_q := \inf \left\{ \frac{\|\nabla u\|_p^p}{\|u\|_q^p} : u \in W^{1,p}_0(\Omega) \setminus \{0\} \right\}$, where $\Omega$ is a bounded and smooth domain of $\mathbb{R}^N$, $1 < p < N$ and $1 \leq q \leq p^* := \frac{Np}{N-p}$. ($\sqrt[2]{\lambda_q}$ is the best constant in the Sobolev immersion $W^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega)$.)

For each $q \in [1, p^*)$ let

$$E_q := \{ u \in W^{1,p}_0(\Omega) : \|u\|_q = 1 \text{ and } \|\nabla u\|_p = \sqrt[2]{\lambda_q} \}$$

denote the set of the $L^2$-normalized extremal functions corresponding to $\lambda_q$.

We prove that the following representation formula

$$\lambda_q = \lambda_1 \exp \left( -p \int_1^q \frac{1}{s^2} \int_\Omega |u_s|^s \log |u_s|^s \, dx \, ds \right)$$

is valid for all $q \in [1, p^*)$, where $u_s \in E_s$.

For this, after presenting some properties of the function $q \in [1, p^*) \mapsto \lambda_q$ (among them the absolute continuity) we verify that

$$\lambda_q' + \lambda_q \left( \frac{p}{q^2} \int_\Omega |u_q|^q \log |u_q|^q \, dx \right) = 0$$

at each point $q$ where the derivative $\lambda_q'$ of $\lambda_q$ exists.

It follows from our results that $\lambda_q$ is differentiable at any $q \in [1, p]$ and, moreover, that $\lambda_q$ is differentiable at $q \in (p, p^*)$ if, and only if, the functional $I_q : W^{1,p}_0(\Omega) \to \mathbb{R}$, defined by

$$I_q(u) := \int_\Omega |u_q|^q \log |u_q|^q \, dx,$$

is constant on $E_q$. Thus, $I_q$ is constant on $E_q$ for almost all $q \in (p, p^*)$ and, in the particular case where $\Omega$ is a ball, $\lambda_q$ is also differentiable at any point of this interval.

*E-mail: grey@mat.ufmg.br. The author was supported by FAPEMIG and CNPq, Brazil.