LECTURES 3-4

SCALAR BVPs WITH MATRIX VARIABLE COEFFICIENT (ANISOTROPIC CASE)

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1. Formulation of the Dirichlet, Neumann, and Robin type BVPs for PDEs with variable matrix coefficients.

2. Localized parametrix. Classes of cut off functions.

3. Harmonic localized parametrix approach: Green's formulas.

4. Properties of localized potentials and reduction to Localized Boundary–domain integral Equations (LBDIE) systems.

3. Equivalence theorems. Comparison with the isotropic case.

4. Some auxiliary theorems.

5. Investigation of the localized LBDIE systems.

O. Chkadua, S. Mikhailov, and D. Natroshvili, Localized boundary - domain singular integral equations based on harmonic parametrix for divergence-form elliptic PDEs with variable matrix coefficients. Integral Equations and Operator Theory, 76, No. 4 (2013), 509–547.

O. Chkadua, S. Mikhailov, and D. Natroshvili, Analysis of some localized boundary-domain integral equations, Journal of Integral Equations and Applications, 21, No. 3 (2009), 407–447.

O. Chkadua, S. Mikhailov, and D. Natroshvili, Analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficient, Part I: Equivalence and invertibility. Journal Integral Equations Appl. 21, No. 4 (2009), 499–542.

O. Chkadua, S. Mikhailov, and D. Natroshvili, Localized boundary - domain singular integral equations based on harmonic parametrix for divergence-form elliptic PDEs with variable matrix coefficients. Integral Equations and Operator Theory, 76, No. 4 (2013), 509–547.

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Vishik-Eskin Theory:

G. Eskin, Boundary Value Problems for Elliptic Pseudodifferential Equations. Transl. of Mathem. Monographs, Amer. Math. Soc., 52, Providence, Rhode Island, 1981.

Formulation of the problems

Consider a uniformly elliptic second order scalar partial differential operator

$$A(x,\partial_x)\,u = \frac{\partial}{\partial x_k}\Big(a_{kj}(x)\,\frac{\partial u}{\partial x_j}\Big), \tag{1}$$

where $a_{kj} = a_{jk} \in C^\infty(\mathbb{R}^3)$, $\mathbf{a} = \big[a_{kj}\big]_{3\times 3}$ is a positive definite matrix, i.e., there are positive constants c_1 and c_2 such that

$$c_1\,|\xi|^2 \le a_{kj}(x)\,\xi_k\,\xi_j \le c_2\,|\xi|^2 \quad \forall\, x \in \mathbb{R}^3, \;\; \forall\, \xi \in \mathbb{R}^3. \tag{2}$$

Formulation of the problems

Consider a uniformly elliptic second order scalar partial differential operator

$$A(x, \partial_x)\, u = \frac{\partial}{\partial x_k}\Big(a_{kj}(x)\,\frac{\partial u}{\partial x_j}\Big), \tag{3}$$

where $a_{kj} = a_{jk} \in C^\infty(\mathbb{R}^3)$, $\mathbf{a} = \big[a_{kj}\big]_{3\times 3}$ is a positive definite matrix, i.e., there are positive constants c_1 and c_2 such that

$$c_1\,|\xi|^2 \le a_{kj}(x)\,\xi_k\,\xi_j \le c_2\,|\xi|^2 \quad \forall\, x \in \mathbb{R}^3, \;\; \forall\, \xi \in \mathbb{R}^3. \tag{4}$$

Let $\Omega^+ \subset \mathbb{R}^3$ be an open bounded domain with a simply connected boundary $\partial\Omega^+ = S \in C^\infty$, $\overline{\Omega^+} = \Omega^+ \cup S$. $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$.

The symbols $\{u\}_S^+ \equiv [u]^+ \equiv u^+$ and $\{u\}_S^- \equiv [u]^- \equiv u^-$ denote one-sided limits (traces) on S from Ω^+ and Ω^-, respectively;

$n = (n_1, n_2, n_3)$ - the outward unit normal vector to S.

The Sobolev-Slobodetskii and Bessel potential spaces:
$W^r(\Omega) = W_2^r = H_2^r$, $r \geqslant 0$, **and** $H^t = H_2^t$, $t \in \mathbb{R}$,

$$H^{1,0}(\Omega^+; A) := \{\, u \in H^1(\Omega^+) \, : \, Au \in H^0(\Omega^+) \,\} \,. \tag{5}$$

The co-normal derivative operators on the surface S for sufficiently smooth functions are defined by the relations

$$T^+(x, n(x), \partial_x)\, u(x) := a_{kj}(x)\, n_k(x)\, \{\partial_j u(x)\}^+, \quad x \in S. \tag{6}$$

The co-normal derivative operator can be extended by continuity to functions $u \in H^{1,0}(\Omega^+; A)$ by Green's first formula

$$\big\langle T^+ u \,,\, v^+ \big\rangle_S = \int_{\Omega^+} \big[v\, Au + a_{kj}(x)\, (\partial_j u)\, (\partial_k v) \big]\, dx, \tag{7}$$

where $v \in H^1(\Omega)$ and $\langle \cdot \,,\, \cdot \rangle_S$ denotes the duality between the mutually adjoint spaces $H^{-\frac{1}{2}}(S)$ and $H^{\frac{1}{2}}(S)$.
Evidently, $T^+ u \in H^{-\frac{1}{2}}(S)$ is well-defined by (7).

Find a function $u \in H^{1,0}(\Omega^+; A)$ satisfying the differential equation

$$A(x, \partial_x)\, u = f \ \text{ in } \ \Omega^+, \qquad f \in H^0(\Omega^+), \tag{8}$$

and one of the following boundary conditions:

Dirichlet condition -

$$\{u\}^+ = \varphi_0 \quad \text{on} \quad S, \qquad \varphi_0 \in H^{\frac{1}{2}}(S); \tag{9}$$

Neumann condition -

$$T^+ u = \psi_0 \quad \text{on} \quad S, \qquad \psi_0 \in H^{-\frac{1}{2}}(S); \tag{10}$$

Robin condition -

$$T^+ u + \kappa\, \{u\}^+ = \psi_1 \quad \text{on} \quad S, \quad \psi_1 \in H^{-\frac{1}{2}}(S), \ \ \kappa \geqslant 0. \tag{11}$$

Equation (8) is understood in the distributional sense, the Dirichlet type boundary condition is understood in the usual trace sense and the Neumann type condition for the co-normal derivative is understood in the generalized functional sense.

Uniqueness

The above formulated Dirichlet and Robin BVPs are uniquely solvable, while the condition

$$\int_{\Omega^+} f(x)\, dx = \langle \psi_0 \, , \, 1 \rangle_S \tag{12}$$

is necessary and sufficient for the Neumann problem to be solvable. A solution of the Neumann problem is defined modulo a constant summand.

Proof follows from Green's first formula

$$\big\langle T^+ u \, , \, u^+ \big\rangle_S = \int_{\Omega^+} \big[u\, A u + a_{kj}(x)\, (\partial_j u)\, (\partial_k u) \big]\, dx \tag{13}$$

and uniform positive definiteness of the matrix $[a_{kj}(x)]_{3\times 3}$.

Classes of localizing functions

DEFINITION. We say $\chi \in X_+^k$ for integer $k \geq 1$ if $\chi(x) \equiv \chi(|x|)$, $\chi \in W_1^k(0,\infty)$, $\chi(0) = 1$, and

$$\sigma_\chi(\omega) > 0 \quad \forall\, \omega \in \mathbb{R}, \tag{14}$$

where

$$\sigma_\chi(\omega) := \begin{cases} \dfrac{\widehat{\chi}_s(\omega)}{\omega} & \text{for } \omega \in \mathbb{R} \setminus \{0\}\,, \\ \displaystyle\int_0^\infty \varrho\, \chi\,(\varrho)\, d\varrho & \text{for } \omega = 0\,, \end{cases} \tag{15}$$

$\widehat{\chi}_s(\omega)$ denotes the sine-transform of the function χ,

$$\widehat{\chi}_s(\omega) := \int_0^\infty \chi\,(\varrho)\, \sin(\varrho\,\omega)\, d\varrho\,. \tag{16}$$

The class X_+^k is defined in terms of the sine-transform. The following lemma provides an easily verifiable sufficient condition for non-negative non-increasing functions to belong to this class.

LEMMA 1. Let $k \geq 1$. If $\chi \in W_1^k(0,\infty)$, $\chi(0) = 1$, $\chi(\varrho) \geq 0$ for all $\varrho \in (0,\infty)$, and χ is a non-increasing function on $[0,+\infty)$, then $\chi \in X_+^k$.

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Examples for χ with a compact support $\overline{B(0,\varepsilon)}$:

$$\chi_{1k}(x) = \begin{cases} \left[1 - \dfrac{|x|^2}{\varepsilon^2}\right]^k & \text{for} \quad |x| < \varepsilon, \\ 0 & \text{for} \quad |x| \geq \varepsilon, \end{cases} \tag{17}$$

$$\chi_2(x) = \begin{cases} \exp\left[\dfrac{|x|^2}{|x|^2 - \varepsilon^2}\right] & \text{for} \quad |x| < \varepsilon, \\ 0 & \text{for} \quad |x| \geq \varepsilon, \end{cases} \tag{18}$$

$$\chi_{1k} \in X_+^k \cap C^{k-1}(\mathbb{R}^3), \quad \chi_2 \in X_+^\infty \cap C^\infty(\mathbb{R}^3). \tag{19}$$

Below we always assume that $\chi \in X_+^k \cap C^2(\mathbb{R}^3)$ with $k \geq 3$ if not otherwise stated.

Localized parametrix-based operators

Define a harmonic localized parametrix corresponding to the fundamental solution $\Gamma(x) := -[\,4\,\pi\,|x|\,]^{-1}$ of the Laplace operator

$$P(x) \equiv P_{\chi}(x) := \chi(x)\,\Gamma(x) = -\frac{\chi(x)}{4\,\pi\,|x|}, \tag{20}$$

where χ is a localizing function.

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where χ is a localizing function.

For $u, v \in H^{1,0}(\Omega; A)$ the following Green's second identity holds

$$\int_{\Omega^+} \big[\, v\,Au - u\,Av\,\big]\,dx = \big\langle T^+u\,,\,v^+\big\rangle_S - \big\langle T^+v\,,\,u^+\big\rangle_S\,. \qquad \textbf{(22)}$$

Take $u \in C^2(\overline{\Omega^+})$ and $v(x) = P(x-y)$, where y is an arbitrarily fixed interior point in Ω^+.
Evidently $v \in C^2(\overline{\Omega^+_\varepsilon})$, where $\Omega^+_\varepsilon := \Omega^+ \setminus \overline{B(y,\varepsilon)}$ with $\varepsilon > 0$, such that the ball $\overline{B(y,\varepsilon)} \subset \Omega^+$, and thus we can write Green's second identity for the region Ω^+_ε:

$$\int_{\Omega_\varepsilon^+} \left[P(x-y)\ A(x,\partial_x)u(x) - u(x)\ A(x,\partial_x)P(x-y) \right] dx =$$

$$= \int_{S\cup\Sigma_\varepsilon} \left[P(x-y)T^+u(x) - \{T(x,\partial_x)P(x-y)\}u^+(x) \right] dS_x. \quad \textbf{(G}\varepsilon\textbf{)}$$

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$$= \int_{S\cup\Sigma_\varepsilon} \big[P(x-y)T^+u(x) - \{T(x,\partial_x)P(x-y)\}u^+(x)\big]\,dS_x. \quad \textbf{(G}\varepsilon\textbf{)}$$

Here $A(x,\partial_x)P(x-y)$ generates a Cauchy singular kernel

$$A(x,\partial_x)P(x-y) = \beta(x)\ \delta(x-y) + \text{v.p.}\, A(x,\partial)P(x-y), \quad \textbf{(23)}$$

with $\beta(x) = 3^{-1}\,[\,a_{11}(x) + a_{22}(x) + a_{33}(x)\,]$,

$$\text{v.p.}\, A(x,\partial)P(x-y) = \text{v.p.}\Big[-\frac{a_{kj}(x)}{4\pi}\frac{\partial^2}{\partial x_k \partial x_j}\frac{1}{|x-y|}\Big] + \quad \textbf{(24)}$$

$$+ R(x,y),$$

$$R(x,y) = \mathcal{O}(|x-y|^{-2}). \quad \textbf{(25)}$$

REMARK. If $a_{kj}(x) = a(x)\delta_{kj}$, then the singular part in (24) vanishes and $A(x,\partial_x)P(x-y)$ becomes a weakly singular kernel. This is a principal difference between the isotropic and anisotropic cases.

By direct calculations one can deduce

$$\lim_{\varepsilon\to 0}\int_{\Sigma_\varepsilon} P(x-y)T(x,\partial_x)u(x)\,dS_x = 0, \tag{26}$$

$$\lim_{\varepsilon\to 0}\int_{\Sigma_\varepsilon} \{T(x,\partial_x)P(x-y)\}\,u(x)\,dS_x = -\beta(y)\,u(y). \tag{27}$$

Introduce the singular integral operator

$$\begin{aligned}\mathcal{N}\,u(y) &:= \ \text{v.p.}\int_{\Omega^+} [\,A(x,\partial_x)P(x-y)\,]\,u(x)\,dx\\ &:= \lim_{\varepsilon\to 0}\int_{\Omega^+_\varepsilon} [\,A(x,\partial_x)P(x-y)\,]\,u(x)\,dx. \qquad \text{(SIO)}\end{aligned}$$

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$$\lim_{\varepsilon\to 0}\int_{\Sigma_\varepsilon} P(x-y)T(x,\partial_x)u(x)\,dS_x = 0, \tag{26}$$

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$$:= \lim_{\varepsilon\to 0}\int_{\Omega^+_\varepsilon} [\,A(x,\partial_x)P(x-y)\,]\,u(x)\,dx. \tag{SIO}$$

Passing to the limit in Green's second formula for the domain Ω_ε as $\varepsilon \to 0$, we obtain **the localized harmonic parametrix-based Green's third identity:**

$$\beta(y)u(y)+\mathcal{N}u(y)-V(T^+u)(y)+W(u^+)(y)=\mathcal{P}\big(Au\big)(y), \tag{G3}$$

$$V(g)(y) := -\int_S P(x-y)\, g(x) dS_x = \int_S \frac{\chi(|x-y|)}{4\pi|x-y|}\, g(x) dS_x, \quad \textbf{(28)}$$

$$W(g)(y) := -\int_S \big[\, T(x, n(x), \partial_x)\, P(x-y) \,\big]\; g(x)\, dS_x, \qquad \textbf{(29)}$$

$$\mathcal{P}(h)(y) := \int_{\Omega^+} P(x-y)\, h(x) dx = -\int_{\Omega^+} \frac{\chi(|x-y|)}{4\pi|x-y|}\, h(x) dx. \quad \textbf{(30)}$$

Due to the density of $C^2(\overline{\Omega^+})$ in $H^{1,0}(\Omega^+; A)$, Green's third identity (G3) is valid also for $u \in H^{1,0}(\Omega^+; A)$.

If the domain of integration is $\mathbb{R}^3$ we use the notation $\mathbf{P}$ and $\mathbf{N}$ for $\mathcal{P}$ and $\mathcal{N}$.

Properties of localized potentials

Introduce the LBDIO generated by the direct values of the localized single and double layer potentials on S:

$$\mathcal{V}\, g(y) := -\int_S P(x-y)\, g(x)\, dS_x, \quad y \in S, \tag{31}$$

$$\mathcal{W}\, g(y) := -\int_S \big[\, T(x, \partial_x)\, P(x-y) \,\big]\, g(x)\, dS_x, \quad y \in S, \tag{32}$$

Note that $\mathcal{V}$ is a weakly singular integral operator (pseudodifferential operator of order -1) and represents a compact perturbation of the harmonic single layer operator;

The operator $\mathcal{W}$ is a singular integral operator (pseudodifferential operator of order 0).

Properties of the localized volume potential

The complete symbol $\mathfrak{S}(\mathrm{P};\xi)$ of the operator P is given by formula:

$$\mathfrak{S}(\mathrm{P};\xi)=\mathcal{F}_{x\to\xi}\Big[-\frac{\chi(x)}{4\pi|x|}\Big]=-\frac{1}{4\,\pi}\int\limits_{\mathbb{R}^3}\frac{\chi(x)}{|x|}e^{i\,x\cdot\xi}dx,\quad \xi\in\mathbb{R}^3 \qquad \textbf{(33)}$$

LEMMA 1. Let $\chi \in X_+^k$ with $k \geq 1$. Then

(i) $\mathfrak{S}(\mathrm{P};\,\cdot) \in C^\infty(\mathbb{R}^3)$ and $\mathfrak{S}(\mathrm{P};\xi) < 0$ for all $\xi \in \mathbb{R}^3$,

(ii) for $\xi \neq 0$ the following equality holds

$$\mathfrak{S}(\mathrm{P};\xi) = \sum_{m=0}^{k^*}\frac{(-1)^{m+1}}{|\xi|^{2m+2}}\chi^{(2m)}(0)-$$
$$-\frac{1}{|\xi|^{k+1}}\int_0^\infty \sin\Big(|\xi|\varrho+\frac{k\pi}{2}\Big)\chi^{(k)}(\varrho)d\varrho, \qquad \textbf{(34)}$$

where k^* is the integer part of $(k-1)/2$.

LEMMA 2. Let $\chi \in X_+^k$ with $k \geq 1$. There exist positive constants c_1 and c_2 such that

$$c_2 \, (1 + |\xi|^2)^{-1} \leq |\mathfrak{S}(\mathrm{P}; \xi)| \leq c_1 \, (1 + |\xi|^2)^{-1} \quad \text{for all} \quad \xi \in \mathbb{R}^3, \qquad \textbf{(35)}$$

and the following operator is invertible

$$\mathrm{P} \; : \; H^s(\mathbb{R}^3) \to H^{s+2}(\mathbb{R}^3) \quad \forall \; s \in \mathbb{R}. \qquad \textbf{(36)}$$

In particular,

$$\mathcal{P} \quad : \quad H^0(\Omega^+) \to H^2(\Omega^+).$$

Properties of the localized layer potentials

The localized single layer potential can be represented in terms of the localized volume potential,

$$V(\psi)(y) = -\int\limits_S P(x-y)\,\psi(x)\,dS_x = -\langle \gamma_S P(\cdot - y), \psi \rangle_S =$$

$$= -\,\langle P(\cdot - y), \gamma_S^* \psi \rangle_{\mathbb{R}^3} = -\mathbf{P}\big(\gamma_S^* \psi\big)(y), \qquad (37)$$

where $\gamma_S^* = \delta_S$ denotes the operator adjoint to the trace operator

$$\gamma_S : H^t(\mathbb{R}^3) \to H^{t-\frac{1}{2}}(S), \quad t > 1/2, \qquad (38)$$

and possesses the following mapping property

$$\gamma_S^* : H^{\frac{1}{2}-t}(\partial\Omega) \to H_S^{-t}(\mathbb{R}^3), \quad t > 1/2, \qquad (39)$$

where the space H_S^{-t} consists of distributions from $H^{-t}(\mathbb{R}^3)$, whose supports belong to $S = \partial\Omega$, i.e. $\operatorname{supp} \gamma_S^* \subset S$.

In turn, the localized double layer potential can also be represented in terms of the localized single layer potential,

$$W(\varphi)(y) = -\int_S \big[T(x,\partial_x)\,P(x-y)\big]\;\varphi(x)\,dS_x =$$

$$= -\int_S \big[\,a_{kj}(x)\,n_k(x)\,\partial_{x_j}P(x-y)\,\big]\,\varphi(x)\,dS_x$$

$$= -\partial_{y_j}\,V(a_{kj}\,n_k\,\varphi)(y)\,,\quad y\in\mathbb{R}^3\setminus S\,. \tag{40}$$

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$$= -\int_S \big[\,a_{kj}(x)\,n_k(x)\,\partial_{x_j}P(x-y)\,\big]\,\varphi(x)\,dS_x$$

$$= -\partial_{y_j}\,V(a_{kj}\,n_k\,\varphi)(y)\,,\quad y\in\mathbb{R}^3\setminus S\,. \tag{41}$$

LEMMA 3. **The following jump relations hold on S:**

$$\{V\psi\}^{\pm} = \mathcal{V}\psi, \qquad \psi\in H^{-\frac{1}{2}}(S), \tag{42}$$

$$\{W\varphi\}^{\pm} = \mp\,\mu\,\varphi + \mathcal{W}\varphi, \qquad \varphi\in H^{\frac{1}{2}}(S), \tag{43}$$

where

$$\mu(y) := \frac{1}{2}\,a_{kj}(y)\,n_k(y)\,n_j(y) > 0, \quad y\in S. \tag{44}$$

LEMMA 4. The following operators are continuous

$$V : H^{-\frac{1}{2}}(S) \to H^{1,0}(\Omega^+; \Delta) \tag{45}$$

$$W : H^{\frac{1}{2}}(S) \to H^{1,0}(\Omega^+; \Delta) \tag{46}$$

$$\mathcal{V} : H^{-\frac{1}{2}}(S) \to H^{\frac{1}{2}}(S), \tag{47}$$

$$\mathcal{W} : H^{\frac{1}{2}}(S) \to H^{\frac{1}{2}}(S), \tag{48}$$

ESSENTIAL REMARK: $H^{1,0}(\Omega^+; \Delta) \neq H^{1,0}(\Omega^+; A)$

TWO BASIC LEMMAS

BASIC LEMMA 1. Let $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$, $f \in H^{0}(\Omega^{+})$. If

$$V(\psi)(y) + \mathcal{P}(f)(y) = 0 \;\; \text{in} \;\; \Omega^{+},$$

then $\psi = 0$ on $\partial\Omega$ and $f = 0$ in Ω^{+}.

[Lemma 6.3 in

O. Chkadua, S. Mikhailov, and D. Natroshvili, Analysis of some localized boundary-domain integral equations, Journal of Integral Equations and Applications, 21, No. 3 (2009), 407–447.]

Now, let us recall localized Green's third identity

$$\beta(y)u(y)+\mathcal{N}u(y)-V(T^{+}u)(y)+W(u^{+})(y)=\mathcal{P}\big(Au\big)(y) \quad \textbf{(49)}$$

and consider the following counterpart of relation (49):

$$\beta(y)u(y)+\mathcal{N}u(y)-V(\psi)(y)+W(\varphi)(y)=F(y)+\mathcal{P}(f)(y), \quad \textbf{(50)}$$

where

$$\psi\in H^{-\frac{1}{2}}(S),\;\; \varphi\in H^{\frac{1}{2}}(S),\;\; F\in H^{1,0}(\Omega^{+};\Delta),\;\; f\in H^{0}(\Omega^{+}) \quad \textbf{(51)}$$

BASIC LEMMA 2. If $u \in H^1(\Omega^+)$ solves equation (50), then $u \in H^{1,0}(\Omega^+, A)$.

COROLLARY 5. If $\chi \in X^3$, then the following operator is bounded

$$\beta\,\mathrm{I}+\mathcal{N} \,:\, H^{1,0}(\Omega^{+},A) \to H^{1,0}(\Omega^{+};\Delta).$$

The singular operator

$$\mathrm{N}\,u(y) := \text{v.p.} \int_{\mathbb{R}^3} [\,A(x,\partial_x)P(x-y)\,]\,u(x)\,dx\,,$$

can be represented as

$$\mathrm{N}\,u(y) = -\beta(y)\,u(y) + \partial_l\,\mathbf{P}\big(a_{kl}\,\partial_k u\big)(y), \quad \forall\, y \in \mathbb{R}^3, \tag{52}$$

and using the mapping properties of the operator $\mathbf{P}$, we deduce that the following operator is continuous

$$\mathrm{N} \;:\; H^s(\mathbb{R}^3) \to H^s(\mathbb{R}^3), \quad s \in \mathbb{R}.$$

Denote by E_0 the extension operator by zero from Ω^+ onto Ω^-. For a function $u \in H^1(\Omega^+)$ we have (note that $E_0 u \notin H^1(\mathbb{R}^3)$!!!)

$$\big(\mathcal{N}\,u\big)(y) = \big(\mathrm{N}E_0 u\big)(y) \quad \text{for} \;\; y \in \Omega^+. \tag{53}$$

Due to (52), this implies continuity of the operator

$$r_{_{\Omega^+}}\mathrm{N}\,E_0 \;:\; H^1(\Omega^+) \to H^1(\Omega^+). \tag{54}$$

Rewrite Green's third formula for $u \in H^{1,0}(\Omega^+; A)$ in a form more convenient for our further analysis:

$$[\,\beta\,\mathrm{I} + \mathrm{N}\,]\,E_0 u\,(y) - V(T^+u)(y) + W(u^+)(y) = \\ = \mathcal{P}\big(A(x, \partial_x)u\big)(y), \quad y \in \Omega^+, \qquad \textbf{(55)}$$

where $\mathbf{I}$ stands for the identity operator.
The trace of equation (55) on S reads as:

$$\mathrm{N}^+\,E_0 u - \mathcal{V}(T^+u) + (\beta - \mu)\,u^+ + \mathcal{W}(u^+) = \\ = \mathcal{P}^+\big(A(x, \partial_x)u\big) \ \text{on} \ S. \qquad \textbf{(56)}$$

where $\mathrm{N}^+w := \{\mathrm{Nw}\}^+_{\mathrm{S}}$ and $\mathcal{P}^+w := \{\mathcal{P}w\}^+_S$.

(55) and (56) are basic equations for the LBDIE method.

Reduction to LBDIE systems and equivalence theorems

LBDIE system for the Dirichlet problem:

$$A(x,\partial_x)\,u = f \ \text{ in } \ \Omega^+, \qquad f\in H^0(\Omega^+), \tag{57}$$

$$\{u\}^+ = \varphi_0 \ \text{ on } \ S, \qquad \varphi_0\in H^{\frac{1}{2}}(S); \tag{58}$$

Equations (55) and (56) can be rewritten as follows

$$[\,\beta\,\mathrm{I}+\mathrm{N}\,]\,E_0 u - V(\psi) = \mathcal{P}(f) - W(\varphi_0) \ \text{ in } \ \Omega^+, \tag{59}$$

$$\mathrm{N}^+\,E_0 u - \mathcal{V}(\psi) = \mathcal{P}^+(f) - (\beta-\mu)\,\varphi_0 - \mathcal{W}(\varphi_0) \ \text{ on } \ S, \tag{60}$$

where $\psi := T^+u \in H^{-\frac{1}{2}}(S)$.
One can consider these relations as the LBDIE system with respect to the segregated unknown functions u and ψ.

EQUIVALENCE THEOREM

THEOREM 6. Let $\chi \in X_+^3$, $\varphi_0 \in H^{\frac{1}{2}}(S)$ and $f \in H^0(\Omega^+)$.

(i) If a function $u \in H^{1,0}(\Omega^+; A)$ solves the Dirichlet BVP (57)-(58), then the solution is unique and the pair

$$(u, \psi) \in H^{1,0}(\Omega^+; A) \times H^{-\frac{1}{2}}(S)$$

with

$$\psi = T^+ u \,, \tag{61}$$

solves the LBDIE system (59)-(60).

(ii) Vice versa, if a pair $(u, \psi) \in H^{1,0}(\Omega^+; A) \times H^{-\frac{1}{2}}(S)$ solves LBDSIE system (59)-(60), then the solution is unique, the function u solves the Dirichlet BVP (8)-(9), and equation (61) holds.

PROOF. **(i)** The first part of the theorem directly follows form Green' third formula.

(ii) Let a pair $(u, \psi) \in H^{1,0}(\Omega^+; A) \times H^{-\frac{1}{2}}(S)$ solve the LBDIE system (59)-(60). Taking the trace of (59) on S and comparing with (60) we get

$$\gamma^+ u = \varphi_0 \quad \text{on} \quad S. \tag{62}$$

Further, by BASIC LEMMA 2, $u \in H^{1,0}(\Omega^+; A)$ and we can write Green's third formula (55) which in view of (62) can be rewritten as

$$[\,\beta\mathbf{I} + \mathbf{N}\,]\, E_0 u - V(T^+ u) = \mathcal{P}\big(A(x, \partial_x) u\big) - W(\varphi_0) \text{ in } \Omega^+. \tag{63}$$

Comparing the relations (59) and (63) we deduce

$$V(T^+ u - \psi) + \mathcal{P}\big(A(x, \partial_x) u - f\big) = 0 \quad \text{in} \quad \Omega^+. \tag{64}$$

Whence by BASIC LEMMA 1, $A(x, \partial_x) u = f$ in Ω^+ and $T^+ u = \psi$ on S.

Thus u solves the Dirichlet BVP and equation (61) holds. ■

Invertibility of the Dirichlet LBDIO

LBDIE system for the Dirichlet problem:

$$[\,\beta\,\mathrm{I} + \mathrm{N}\,]\,E_0 u - V(\psi) = \mathcal{P}(f) - W(\varphi_0) \ \text{ in } \ \Omega^+, \tag{65}$$

$$\mathrm{N}^+ E_0 u - \mathcal{V}(\psi) = \mathcal{P}^+(f) - (\beta - \mu)\,\varphi_0 - \mathcal{W}(\varphi_0) \ \text{ on } \ S, \tag{66}$$

Denote by $\mathfrak{D}$ the localized boundary-domain integral operator generated by the left-hand side expressions in LBDIE system (65)–(66),

$$\mathfrak{D} := \begin{bmatrix} r_{_{\Omega^+}}(\beta\,\mathrm{I} + \mathrm{N})E_0 & -r_{_{\Omega^+}} V \\ \mathrm{N}^+ E_0 & -\mathcal{V} \end{bmatrix}, \tag{67}$$

$$r_{_{\Omega^+}}\mathrm{N}\,E_0 \ : \ H^1(\Omega^+) \to H^1(\Omega^+), \tag{68}$$

$$V \ : \ H^{-\frac{1}{2}}(S) \to H^1(\Omega^+), \tag{69}$$

$$\Downarrow$$

$$\mathfrak{D} \ : H^1(\Omega^+) \times H^{-\frac{1}{2}}(S) \to H^1(\Omega^+) \times H^{\frac{1}{2}}(S)\,. \tag{70}$$

INVERTIBILITY of (70) is quite nontrivial!

The main goal is to show that the equation

$$\mathfrak{D}\begin{bmatrix} u \\ \psi \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{71}$$

i.e., the system

$$r_{_{\Omega^+}}(\beta\,\mathrm{I} + \mathrm{N})\,E_0 u - V(\psi) = F_1 \quad \text{in} \quad \Omega^+, \tag{72}$$

$$\mathrm{N}^+\,E_0 u - \mathcal{V}(\psi) = F_2 \quad \text{on} \quad S, \tag{73}$$

is uniquely solvable in the space $H^1(\Omega^+) \times H^{-\frac{1}{2}}(S)$ for arbitrary $F_1 \in H^1(\Omega^+)$ and $F_2 \in H^{\frac{1}{2}}(S)$.

Invertibility of the operator $\mathfrak{D}$ is shown in several steps.

STEP 1: Fredholm properties of the domain operators

The principal homogeneous symbols of the singular integral operators $\mathbf{N}$ and $\beta\,\mathbf{I}+\mathbf{N}$ read as

$$\mathfrak{S}_0(\mathrm{N};y,\xi)=\frac{A(y,\xi)-\beta\,|\xi|^2}{|\xi|^2},\;\;\forall\,y\in\mathbb{R}^3,\;\forall\,\xi\in\mathbb{R}^3\setminus\{0\},\quad\textbf{(74)}$$

$$\mathfrak{S}_0(\beta\mathrm{I}+\mathrm{N};y,\xi)=\frac{A(y,\xi)}{|\xi|^2}>0,\;\;\forall y\in\mathbb{R}^3,\;\forall\xi\in\mathbb{R}^3\setminus\{0\}.\quad\textbf{(75)}$$

$$A(y,\xi):=a_{kl}(y)\,\xi_k\,\xi_l,\quad\beta(y)=3^{-1}\big[a_{11}+a_{22}+a_{33}\big]\quad\textbf{(76)}$$

These principal homogeneous symbols are even rational homogeneous functions in ξ of order 0.

This plays a crucial role in the study of the operator $\mathfrak{D}$.

Introduce the notation:

$$\mathbf{B} \equiv \boldsymbol{r}_{_{\Omega^+}} (\boldsymbol{\beta}\, \mathbf{I} + \mathbf{N})\, \boldsymbol{E}_0. \tag{77}$$

LEMMA 7. **The operator**

$$\mathbf{B} \;:\; \boldsymbol{H}^1(\boldsymbol{\Omega}^+) \to \boldsymbol{H}^1(\boldsymbol{\Omega}^+) \tag{78}$$

is Fredholm with zero index.

Introduce the notation:

$$\mathrm{B} \equiv r_{_{\Omega^+}} (\beta\, \mathrm{I} + \mathrm{N})\, E_0. \tag{79}$$

LEMMA 7. The operator

$$\mathrm{B} \;:\; H^1(\Omega^+) \to H^1(\Omega^+) \tag{80}$$

is Fredholm with zero index.

Proof. Since the principal homogeneous symbol $\mathfrak{S}_0(\mathrm{B}; y, \xi)$ of the operator B is an even, rational, positive, homogeneous function of order 0 in $\xi \in \mathbb{R}^3 \setminus \{0\}$,

$$\mathfrak{S}_0(\mathrm{B}; y, \xi) = \frac{A(y,\xi)}{|\xi|^2} = \frac{a_{kl}(y)\xi_k\xi_l}{\xi_1^2 + \xi_2^2 + \xi_3^2} > 0, \tag{81}$$

it follows that the factorization index $\varkappa$ of the symbol (81) equals to zero, and due to the general theory of pseudodifferential operators we deduce that the operator (80) is Fredholm for all $s \geq 0$ [G.ESKIN].

To show that **Ind** $\mathrm{B} = 0$ we use the fact that the operators B and

$$\mathrm{B}_t = r_{_{\Omega^+}} [\,(1-t)\,\mathrm{I} + t\,(\beta\,\mathrm{I} + \mathrm{N})\,]\,E_0, \quad t \in [0,1], \qquad \textbf{(82)}$$

are homotopic. Note that $\mathrm{B}_0 = r_{_{\Omega^+}} \mathrm{I} E_0$ is invertible and $\mathrm{B}_1 = \mathrm{B}$. For the principal homogeneous symbol of the operator B_t we have

$$\mathfrak{S}_0(\mathrm{B}_t; y, \xi) = \frac{(1-t)|\xi|^2 + t\, a_{kl}(y)\xi_k\xi_l}{|\xi|^2} > 0.$$

Since $\mathfrak{S}_0(\mathrm{B}_t; y, \xi)$ is rational, even, and homogeneous of order zero in ξ, we conclude that the operator $\mathrm{B}_t : H^s(\Omega^+) \to H^s(\Omega^+)$ is Fredholm for all $s \geq 0$ and for all $t \in [0,1]$.

Therefore **Ind** B_t is the same for all $t \in [0,1]$. Since $\mathrm{B}_0 = \mathrm{I}$ is invertible, we get $\mathrm{Ind}\,\mathrm{B} = \mathrm{Ind}\,\mathrm{B}_1 = \mathrm{Ind}\,\mathrm{B}_t = \mathrm{Ind}\,\mathrm{B}_0 = 0.$ ■

STEP 2: Fredholm properties of the operator $\mathfrak{D}$

LEMMA 8. The operator

$$\mathfrak{D} \ : H^1(\Omega^+) \times H^{-\frac{1}{2}}(S) \to H^1(\Omega^+) \times H^{\frac{1}{2}}(S) \,. \qquad (83)$$

is Fredholm with zero index.

Proof. Recall that

$$\mathfrak{D} := \begin{bmatrix} \mathrm{B} & -r_{\Omega^+} V \\ \mathrm{N}^+ E_0 & -\mathcal{V} \end{bmatrix} . \qquad (84)$$

We apply Vishik-Eskin theory for pseudodifferential operators based on the local principal which states that since the operator $\mathrm{B} = r_{\Omega^+}(\beta\,\mathrm{I} + \mathrm{N})\,E_0$ is Fredholm with zero index, it follows that the operator (83) is Fredholm if the so-called **generalized Šapiro-Lopatinskiĭ condition** holds.

To formulate the Šapiro-Lopatinskiĭ condition we have to introduce two operators Π^+ and Π':

$$\Pi^+(h)(\xi) := \frac{i}{2\pi} \lim_{t\to 0+} \int_{-\infty}^{+\infty} \frac{h(\xi', \eta_3)\, d\eta_3}{\xi_3 + i\, t - \eta_3}\,, \tag{85}$$

$$\Pi'(g)(\xi') = -\frac{1}{2\pi} \int_{\Gamma^-} g(\xi', \zeta)\, d\zeta, \tag{86}$$

$$\xi = (\xi', \xi_3) \in \mathbb{R}^3, \quad \xi' = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}, \tag{87}$$

where $g(\xi', \zeta)$ is a rational function of a complex variable ζ and Γ^- is a contour in the lower ζ-complex half-plane, orientated anticlockwise and enclosing all the poles of the rational function $g(\xi', \zeta)$ with respect to ζ.

Šapiro-Lopatinskiĭ condition for the operator $\mathfrak{D}$:

$$\mathrm{e}(y,\xi') = -\Pi'\left[\frac{\mathfrak{S}_0(\mathrm{N};y,\cdot)}{\mathfrak{S}_0^{(+)}(\mathrm{B};y,\cdot)}\,\Pi^+\left(\frac{\mathfrak{S}_0(\mathrm{P};y,\cdot)}{\mathfrak{S}_0^{(-)}(\mathrm{B};y,\cdot)}\right)\right](\xi') - \mathfrak{S}_0(\mathcal{V};y,\xi') \neq 0, \tag{88}$$

$$\forall\,\xi' = (\xi_1,\xi_2) \neq 0, \quad \forall\, y \in \partial\Omega,$$

$\mathfrak{S}_0(\mathrm{N})$, $\mathfrak{S}_0(\mathrm{B})$, $\mathfrak{S}_0(\mathrm{P})$, and $\mathfrak{S}_0(\mathcal{V})$ are the corresponding principal homogeneous symbols

$\mathfrak{S}_0^{(+)}(\mathrm{B})$ and $\mathfrak{S}_0^{(-)}(\mathrm{B})$ denote the so called "plus" and "minus" factors in the factorization of the symbol $\mathfrak{S}_0(\mathrm{B};y,\xi)$ with respect to the variable ξ_3:

$$\mathfrak{S}_0(\mathrm{B}) = \mathfrak{S}_0^{(+)}(\mathrm{B})\,\mathfrak{S}_0^{(-)}(\mathrm{B}).$$

By direct calculations it is shown that the **Šapiro-Lopatinskiĭ condition** for the operator $\mathfrak{D}$ holds!
Therefore the operator $\mathfrak{D}$ is Fredholm.

Further it is shown that the operator

$$\mathfrak{D}_t := \begin{bmatrix} \mathrm{B} & -r_{\Omega^+}V \\ t\,\mathrm{N}^+E_0 & -\mathcal{V} \end{bmatrix} \qquad \text{with} \quad t \in [0,1]$$

is homotopic to the operator $\mathfrak{D} = \mathfrak{D}_1$ and the Šapiro-Lopatinskiĭ condition for the operator $\mathfrak{D}_t$ is also satisfied for all $t \in [0,1]$, which implies that the operator

$$\mathfrak{D}_t \;: H^1(\Omega^+) \times H^{-\frac{1}{2}}(S) \to H^1(\Omega^+) \times H^{\frac{1}{2}}(S)$$

is Fredholm and has the same index for all $t \in [0,1]$.

On the other hand, since the upper triangular matrix operator $\mathfrak{D}_0$ has zero index, it follows that

$$\mathrm{Ind}\,\mathfrak{D} = \mathrm{Ind}\,\mathfrak{D}_1 = \mathrm{Ind}\,\mathfrak{D}_t = \mathrm{Ind}\,\mathfrak{D}_0 = 0\,.$$

THEOREM 9. The operator

$$\mathfrak{D} \; : H^1(\Omega^+) \times H^{-\frac{1}{2}}(S) \to H^1(\Omega^+) \times H^{\frac{1}{2}}(S) \,. \tag{89}$$

is invertible.

Proof. Since the operator (89) is Fredholm with zero index it remains to show that the null space of the operator $\mathfrak{D}$ is trivial.
Assume that $U = (u, \psi)^\top \in H^1(\Omega^+) \times H^{-\frac{1}{2}}(S)$ is a solution to the homogeneous equation $\mathfrak{D}\, U = 0$, i.e.

$$r_{\Omega^+} (\beta\, \mathrm{I} + \mathrm{N})\, E_0 u - V(\psi) = 0 \;\text{ in }\; \Omega^+, \tag{90}$$

$$\mathrm{N}^+ E_0 u - \mathcal{V}(\psi) = 0 \;\text{ on }\; S, \tag{91}$$

By BASIC LEMMA 2, from (90) it follows that $u \in H^{1,0}(\Omega^+, A)$.

Further, by EQUIVALENCE THEOREM, we conclude that u solves the homogeneous Dirichlet problem and the relation $T^+ u = \psi$ holds on S, which implies that $U = (u, \psi)^\top$ is zero vector. Thus the null space of the operator $\mathfrak{D}$ is trivial in the class $H^1(\Omega^+) \times H^{-\frac{1}{2}}(S)$. Consequently, the operator $\mathfrak{D}$ is invertible. ∎

COROLLARY 10. The operator

$$\mathfrak{D}\ : H^{1,0}(\Omega^+, A) \times H^{-\frac{1}{2}}(S) \to H^{1,0}(\Omega^+, \Delta) \times H^{\frac{1}{2}}(S) \qquad \textbf{(92)}$$

is invertible.

For a localizing function χ of infinite smoothness the following regularity result holds.

THEOREM 11. Let $\chi \in X_+^\infty$ and $r > 0$. Then the operator

$$\mathfrak{D}\ : H^{r+1}(\Omega^+) \times H^{r-\frac{1}{2}}(S) \to H^{r+1}(\Omega^+) \times H^{r+\frac{1}{2}}(S) \qquad \textbf{(93)}$$

is invertible.

THANK YOU!