LECTURE 5

APPLICATIONS OF BDIE METHOD: ACOUSTIC SCATTERING BY INHOMOGENEOUS ANISOTROPIC OBSTACLES

DAVID NATROSHVILI

Georgian Technical University

Tbilisi, GEORGIA

1. **Formulation of the corresponding transmission problem**
2. **Basic integral relations**
3. **Equivalent reduction to LBDIE system**
4. **Uniqueness theorem**
5. **Investigation of the LBDIO and Existence result**

1. **Formulation of the corresponding transmission problem.**
2. **Basic integral relations.**
3. **Equivalent reduction to LBDIE system.**
4. **Uniqueness theorem.**
5. **Investigation of the LBDIO and Existence result**

O. Chkadua, S. Mikhailov, and D. Natroshvili, Localized boundary-domain integral equations for acoustic scattering by inhomogeneous anisotropic obstacles. (Prepared for publication)

O. Chkadua, S. Mikhailov, and D. Natroshvili, Localized boundary-domain singular integral equations based on harmonic parametrix for divergence-form elliptic PDEs with variable matrix coefficients. Integral Equations and Operator Theory, 76, No. 4 (2013), 509–547.

O. Chkadua, S. Mikhailov, and D. Natroshvili, Analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficient, Part I: Equivalence and invertibility. J. Integral Equations Appl. 21(2009), No. 4, 499–542.

D. Colton and R. Kress, Inverse Acoustic and Electromagnetic Scattering Theory. Third Edition, Springer-Verlag, Berlin, 2013.

P. Werner, Zur mathematischen Theorie akustischer Wellenfelder. Arch. Ration. Mech. Anal., 6 (1961), 231–260.

P.A. Martin, Acoustic scattering by inhomogeneous obstacles. SIAM Journal of Applied Mathematics, 64, No. 1 (2003), 297–308.

INTRODUCTION

We consider the time-harmonic acoustic wave scattering by a bounded anisotropic inhomogeneity embedded in an unbounded anisotropic homogeneous medium.

We assume that the material parameters are functions of position within the inhomogeneous bounded obstacle.

The problem is formulated as a transmission problem (TP) for a second order elliptic partial differential equation with variable coefficients

$$A_2(x,\partial_x)u(x) \equiv \partial_{x_k}\big[a^{(2)}_{kj}(x)\partial_{x_j}u(x)\big] + \omega^2\kappa(x)u(x) = f_2 \qquad \textbf{(1)}$$

in the inhomogeneous anisotropic region Ω_2 and for the "anisotropic" Helmoltz type equation with constant coefficients

$$A_1(\partial_x)u(x) \equiv \partial_{x_k}\big[a^{(1)}_{kj}\,\partial_{x_j}u(x)\big] + \omega^2\,u(x) = f_1 \qquad \textbf{(2)}$$

in the unbounded homogeneous region Ω_1.

Since the material parameters $a_{kj}^{(q)}$ and the refractive index κ are assumed to be discontinuous across the interface $S = \partial\Omega_1 = \partial\Omega_2$ between the inhomogeneous interior and homogeneous exterior regions, there are given standard transmission conditions relating the interior and exterior traces of the sought for wave functions and their co-normal derivatives on S.

Since the material parameters $a_{kj}^{(q)}$ and the refractive index κ are assumed to be discontinuous across the interface $S = \partial\Omega_1 = \partial\Omega_2$ between the inhomogeneous interior and homogeneous exterior regions, there are given standard transmission conditions relating the interior and exterior traces of the sought for wave functions and their co-normal derivatives on S.

In a particular case of isotropic inhomogeneity, when

$$a_{kj}^{(2)}(x) = \delta_{kj}, \qquad a_{kj}^{(1)} = \delta_{kj},$$

i.e., when

$$A_2(x, \partial_x) = \Delta + \omega^2 \kappa(x), \qquad A_1(\partial_x) = \Delta + \omega^2,$$

the similar transmission problems, is well investigated in the literature (e.g. [Colton, Kress](2013): "Lippmann–Schwinger Equation" - Fredholm-Riesz type integral equation).

Another particular case of isotropic inhomogeneity, when

$$a_{kj}^{(2)}(x) = a(x)\,\delta_{kj}, \qquad a_{kj}^{(1)} = \delta_{kj},$$

i.e., when

$$A_2(x,\partial_x)u(x) = \partial_{x_k}\big[a(x)\partial_{x_k}u(x)\big] + \omega^2\kappa(x)u(x), \tag{3}$$

$$A_1(\partial_x)u(x) = \Delta u(x) + \omega^2 u(x), \tag{4}$$

by the indirect boundary-domain integral equation method is investigated by P.Werner (1960).
P.Werner reduces the problem to the Fredholm-Riesz type integral equations system and proves its unique solvability.

Another particular case of isotropic inhomogeneity, when

$$a_{kj}^{(2)}(x) = a(x)\,\delta_{kj}, \qquad a_{kj}^{(1)} = \delta_{kj},$$

i.e., when

$$A_2(x,\partial_x)u(x) = \partial_{x_k}\big[a(x)\partial_{x_k}u(x)\big] + \omega^2\kappa(x)u(x), \tag{3}$$

$$A_1(\partial_x)u(x) = \Delta u(x) + \omega^2 u(x), \tag{4}$$

by the indirect boundary-domain integral equation method is investigated by P.Werner (1960).

P.Werner reduces the problem to the Fredholm-Riesz type integral equations system and proves its unique solvability.

The same problem for isotropic inhomogeneity by the direct method is considered by P.Martin (2003) (using essentially the existence results obtained by P.Werner).

Our goal is to consider the above described wave scattering problems for general inhomogeneous anisotropic case, applying the method of Localized Boundary-Domain Integral Equations.

FORMULATION OF THE TRANSMISSION PROBLEM

Let $\Omega^+ = \Omega_2$ be a bounded domain in $\mathbb{R}^3$ with a simply connected boundary $\partial\Omega^+ = S$, $\overline{\Omega^+} = \Omega^+ \cup S$, and $\Omega_1 = \Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$. For simplicity, we assume that $S \in C^\infty$ if not otherwise stated.

$n = (n_1, n_2, n_3)$ denotes the unit normal vector to S directed outward with respect to the domain Ω^+.

We assume that the propagation region of time harmonic acoustic waves is all of $\mathbb{R}^3$ which consists of an anisotropic inhomogeneous part $\Omega_2 := \Omega^+$ and a anisotropic homogenous one $\Omega_1 := \Omega^-$.

Acoustic wave propagation in Ω_1 and Ω_2 is governed by the uniformly elliptic second order scalar partial differential equations

$$A_1(\partial_x)u_1(x) \equiv \partial_{x_k}\Big(a_{kj}^{(1)}\partial_{x_j}u_1(x)\Big) + \omega^2 u_1(x) = 0, \quad x \in \Omega_1, \qquad (5)$$

$$A_2(x,\partial_x)u_2(x) \equiv \partial_{x_k}\Big(a_{kj}^{(2)}(x)\partial_{x_j}u_2(x)\Big) + \omega^2\kappa(x)u_2(x) =$$
$$= f(x), \quad x \in \Omega_2; \qquad (6)$$

$u_1,\ u_2$ − wave amplitudes, $\omega \in \mathbb{R}$ − wave number,
$\kappa(x) \geqslant 0$ − refractive index, $f \in L_2(\Omega_2)$ − a given function.

$$a_{kj}^{(2)}, \kappa \in C^2(\overline{\Omega}_2), \quad a_{kj}^{(q)} = a_{jk}^{(q)}, \quad j,k = 1,2,3, \quad q = 1,2, \qquad (7)$$

The matrices $\mathbf{a}^{(q)} = [a_{kj}^{(q)}]_{3\times 3}$ are uniformly positive definite, i.e., there are positive constants c_1 and c_2 such that

$$c_1\,|\xi|^2 \le a_{kj}^{(q)}(x)\xi_k\,\xi_j \le c_2|\xi|^2, \quad x \in \overline{\Omega}_q, \ \xi \in \mathbb{R}^3, \ q = 1,2. \qquad (8)$$

In the unbounded region $\Omega_1 = \Omega^-$ we have a total wave field

$$u = u^{tot} = u^{ins} + u^{sc},$$

where u^{ins} is a wave motion initiating known incident field and $u_1 = u^{sc}$ is a radiating unknown scattered field.

In the unbounded region $\Omega_1 = \Omega^-$ we have a total wave field

$$u = u^{tot} = u^{ins} + u^{sc},$$

where u^{ins} is a wave motion initiating known incident field and $u_1 = u^{sc}$ is a radiating unknown scattered field.

The co-normal derivative operators read as

$$\{T_1(x, \partial_x)v(x)\}^- := \{a_{kj}^{(1)}\, n_k(x)\, \partial_{x_j} v(x)\}^-, \quad x \in S, \tag{9}$$

$$\{T_2(x, \partial_x)v(x)\}^+ := \{a_{kj}^{(2)}(x)\, n_k(x)\, \partial_{x_j} v(x)\}^+, \quad x \in S; \tag{10}$$

$u^\pm \equiv \{u\}^\pm$, $T_j^\pm u \equiv \{T_j\, u\}^\pm -$ one sided traces on S from $\Omega^\pm$.

SOMMERFELD RADIATION CONDITIONS

Denote by S_ω the characteristic surface (ellipsoid) associated with the operator $A_1(\partial) = a^{(1)}_{kj}\partial_k\partial_j$,

$$\Phi_1(\xi,\omega) := a^{(1)}_{kj}\xi_k\,\xi_j - \omega^2 = 0, \quad \xi \in \mathbb{R}^3.$$

For an arbitrary vector $\eta \in \mathbb{R}^3$ with $|\eta| = 1$ there exists only one point $\xi(\eta) \in S_\omega$ such that the outward unit normal vector $n(\xi(\eta))$ to S_ω at the point $\xi(\eta)$ has the same direction as η, i.e., $n(\xi(\eta)) = \eta$.
Note that $\xi(-\eta) = -\xi(\eta) \in S_\omega$ and $n(-\xi(\eta)) = -\eta$.
It can be easily verified that

$$\xi(\eta) = \omega\,(\mathrm{b}\eta\cdot\eta)^{-\frac{1}{2}}\ \mathrm{b}\eta, \tag{11}$$

where $\mathrm{b} := [\mathrm{a}^{(1)}]^{-1}$ is the matrix inverse to $\mathrm{a}^{(1)} := [a^{(1)}_{kj}]_{3\times 3}$.

SOMMERFELD CLASS $Z(\Omega^-)$ associated with the operator $A_1(\partial)$:

A complex valued function w belongs to $Z(\Omega^-)$ if for sufficiently large $|x| \gg 1$ the following radiation conditions hold

$$w(x) = \mathcal{O}(|x|^{-1}), \tag{12}$$

$$\partial_k w(x) - i\xi_k(\eta)w(x) = \mathcal{O}(|x|^{-2}), \; k = 1, 2, 3, \tag{13}$$

where $\xi(\eta) \in S_\omega$ corresponds to the vector $\eta = x/|x|$.

LEMMA 1. [Analogue of the Rellich-Vekua lemma, 1943] Let w be a solution of the homogeneous equation $A_1(\partial_x)w = 0$ in Ω^- and let

$$\lim_{R\to+\infty} \int_{\Sigma_R} |w(x)|^2 \, d\Sigma_R = 0, \tag{14}$$

where Σ_R is the sphere centered at the origin and radius R.

Then $w = 0$ in Ω^-.

TRANSMISSION PROBLEM:

Find complex valued functions

$$u_2 \in H^{1,0}(\Omega^+, A_2), \qquad u_1 \in H^{1,0}_{loc}(\Omega^-, A_1) \cap Z(\Omega^-)$$

satisfying the differential equations

$$A_1(\partial_x) u_1(x) = 0 \quad \text{in} \quad \Omega_1, \tag{15}$$

$$A_2(x, \partial_x) u_2(x) = f(x) \quad \text{in} \quad \Omega_2, \tag{16}$$

and the transmission conditions on the interface S

$$u_2^+ - u_1^- = \varphi_0 \quad \text{on} \quad S, \tag{17}$$

$$T_2^+ u_2 - T_1^- u_1 = \psi_0 \quad \text{on} \quad S, \tag{18}$$

where

$$\varphi_0 := \{u^{inc}\}^- \in H^{\frac{1}{2}}(S), \quad \psi_0 := \{T_1 u^{inc}\}^- \in H^{-\frac{1}{2}}(S), \tag{19}$$

$$f \in H^0(\Omega^+). \tag{20}$$

Basic integral relations in the bounded domain $\Omega_2 = \Omega^+$

Localized harmonic parametrix

$$P(x) \equiv P_\chi(x) := -\frac{\chi(x)}{4\,\pi\,|x|}, \qquad \chi \in X_+^k, \quad k \geq 3. \tag{21}$$

Green's third formula for $u_2 \in H^{1,0}(\Omega^+, A_2)$

$$\beta(y)\,u_2(y) + \mathcal{N}_2\,u_2(y) - V_2(T_2^+ u_2)(y) + \\ + W_2(u_2^+)(y) = \mathcal{P}_2\big(A_2 u_2\big)(y), \quad y \in \Omega^+, \tag{22}$$

$$\beta(x) = \frac{1}{3}\,[\,a_{11}^{(2)}(x) + a_{22}^{(2)}(x) + a_{33}^{(2)}(x)\,], \tag{23}$$

$$\mathcal{N}_2\,v(y) := \text{v.p.}\int_{\Omega^+} [A_2(x, \partial_x) P(x-y)] v(x)\,dx, \tag{24}$$

V_2, W_2 and $\mathcal{P}_2$ are the localized single layer, double layer and Newtonian volume potentials

$$V_2(g)(y) := -\int_S P(x-y)\,g(x)\,dS_x, \tag{25}$$

$$W_2(g)(y) := -\int_S \big[\,T_2(x,\partial_x)\,P(x-y)\,\big]\;g(x)\,dS_x, \tag{26}$$

$$\mathcal{P}_2(h)(y) := \int_{\Omega^+} P(x-y)\,h(x)\,dx. \tag{27}$$

The principal homogeneous symbol $\mathfrak{S}_0(\mathcal{N}_2; y,\xi)$ is rational in ξ

$$\mathfrak{S}_0(\mathcal{N}_2; y,\xi) = \frac{a^{(2)}_{kl}(y)\,\xi_k\,\xi_l}{|\xi|^2} - \beta(y) = \frac{A_2(y,\xi)}{|\xi|^2} - \beta(y)\,, \tag{28}$$
$$y \in \overline{\Omega^+},\quad \xi \in \mathbb{R}^3,$$

If $a^{(2)}_{kl}(y) = a^{(2)}(y)\delta_{kl}$, then $\mathfrak{S}_0(\mathcal{N}_2; y,\xi) = 0$ and the operator $\mathcal{N}_2$ becomes weakly singular integral operator.

The interior trace of Green's third formula on S reads as

$$\mathcal{N}_2^+ u_2 - \mathcal{V}_2(T_2^+ u_2) + (\beta - \mu + \mathcal{W}_2) u_2^+ = \mathcal{P}_2^+\big(A_2 u_2\big) \quad \text{on} \quad S, \tag{29}$$

where

$$\mu(y) := \frac{1}{2}\, a_{kj}^{(2)}(y)\, n_k(y)\, n_j(y) > 0, \quad y \in S, \tag{30}$$

$$\mathcal{V}_2\, g(y) := -\int_S P(x-y)\, g(x)\, dS_x, \quad y \in S, \tag{31}$$

$$\mathcal{W}_2\, g(y) := -\int_S \big[\, T_2(x, \partial_x)\, P(x-y) \,\big]\, g(x)\, dS_x, \quad y \in S, \tag{32}$$

$$\mathcal{N}_2^+\, u_2 := \{\mathcal{N}_2\, u_2\}_S^+, \qquad \mathcal{P}_2^+\big(A_2 u_2\big) := \{\mathcal{P}_2\big(A_2 u_2\big)\}^+. \tag{33}$$

LEMMA 2. Let

$$\Phi_2 \in H^{1,0}(\Omega^+, \Delta), \quad \psi_2 \in H^{-\frac{1}{2}}(S), \quad \varphi_2 \in H^{\frac{1}{2}}(S). \tag{34}$$

Moreover, let $u_2 \in H^1(\Omega^+)$ and the following equation hold in Ω^+

$$\beta(y)\, u_2(y) + \mathcal{N}_2\, u_2(y) - V_2(\psi_2)(y) + W_2(\varphi_2)(y) = \Phi(y). \tag{35}$$

Then $u \in H^{1,0}(\Omega^+, A_2)$.

Basic relations in unbounded domain $\Omega_1 = \Omega^-$

For any radiating solution $u_1 \in H^{1,0}_{loc}(\Omega^-, A_1) \cap Z(\Omega^-)$ of homogeneous equation $A_1(\partial)u_1 = 0$ there holds the following Green's third formula:

$$u_1(y) + V_1(T_1^- u_1)(y) - W_1(u_1^-)(y) = 0, \quad y \in \Omega^-, \tag{36}$$

where

$$V_1(g)(y) := -\int_S \gamma(x-y)\, g(x)\, dS_x, \quad y \in \mathbb{R}^3 \setminus S, \tag{37}$$

$$W_1(g)(y) := -\int_S [T_1(x,\partial_x)\gamma(x-y)]\, g(x)\, dS_x, \quad y \in \mathbb{R}^3 \setminus S. \tag{38}$$

$\gamma(x,\omega)$ is a radiating fundamental function of the operator $A_1(\partial_x)$:

$$\gamma(x,\omega) = -\frac{\exp\{i\omega(\mathrm{b}x\cdot x)^{1/2}\}}{4\pi(\det \mathrm{a})^{1/2}(\mathrm{b}x\cdot x)^{1/2}}, \quad \mathrm{b} = \mathrm{a}^{-1}, \ \mathrm{a} = [a^{(1)}_{kj}]_{3\times 3}. \tag{39}$$

Properties of radiating potentials

(i) **Mapping properties**

$$\begin{aligned} V_1 &: H^{-\frac{1}{2}}(S) \to H^1_{loc}(\Omega^-, A_1) \cap Z(\Omega^-), \\ W_1 &: H^{\frac{1}{2}}(S) \to H^1_{loc}(\Omega^-, A_1) \cap Z(\Omega^-). \end{aligned} \qquad (40)$$

(ii) **Jump relations for** $h \in H^{-\frac{1}{2}}(S)$ **and** $g \in H^{\frac{1}{2}}(S)$:

$$\{V_1(h)\}^+ = \{V_1(h)\}^- = \mathcal{V}_1(h) \quad \text{on} \quad S, \qquad (41)$$

$$\{T_1 V_1(h)\}^\pm = (\pm\, 2^{-1} I + \mathcal{W}_1')h \quad \text{on} \quad S, \qquad (42)$$

$$\{W_1(g)(y)\}^\pm = (\mp\, 2^{-1} I + \mathcal{W}_1)g \quad \text{on} \quad S, \qquad (43)$$

$$T_1^+ W_1(g) = T_1^- W_1(g) =: \mathcal{L}_1(g) \quad \text{on} \quad S, \qquad (44)$$

where I stands for the identity operator, and

$$\mathcal{V}_1(h)(y) := -\int_S \gamma(x-y)\, h(x)\, dS_x, \quad y \in S, \tag{45}$$

$$\mathcal{W}_1(g)(y) := -\int_S [T_1(x,\partial_x)\gamma(x-y)]\, g(x)\, dS_x, \quad y \in S, \tag{46}$$

$$\mathcal{W}'_1(h)(y) := -\int_S [T_1(y,\partial_y)\gamma(x-y)]\, h(x)\, dS_x, \quad y \in S. \tag{47}$$

(iii) The operators

$$\mathcal{V}_1 \;:\; H^{-\frac{1}{2}}(S) \to H^{\frac{1}{2}}(S), \tag{48}$$

$$\mathcal{W}_1 \;:\; H^{\frac{1}{2}}(S) \to H^{\frac{1}{2}}(S), \tag{49}$$

$$\mathcal{W}'_1 \;:\; H^{-\frac{1}{2}}(S) \to H^{-\frac{1}{2}}(S), \tag{50}$$

$$\mathcal{L}_1 \;:\; H^{\frac{1}{2}}(S) \to H^{-\frac{1}{2}}(S), \tag{51}$$

are continuous.
The operators (49)-(51) are compact, $\mathcal{V}_1$ and $\mathcal{L}_1$ are pseudodifferential operators of order -1 and 1 respectively.

Let α be a positive constant and define the operators $\mathcal{K}_1$ and $\mathcal{M}_1$:

$$\mathcal{K}_1 \;:= 2^{-1}\, I + \mathcal{W}_1' \; - i\,\alpha\,\mathcal{V}_1 \;\sim\; \{[T_1 - i\alpha]V_1(g)\}^+, \qquad (52)$$

$$\mathcal{M}_1 := \mathcal{L}_1 - i\,\alpha\,(-2^{-1}\, I + \mathcal{W}_1) \;\sim\; \{[T_1 - i\alpha]W_1(g)\}^+. \qquad (53)$$

For a solution $u_1 \in H^{1,\,0}_{loc}(\Omega^-, A_1) \cap Z(\Omega^-)$ of equation $A_1 u_1 = 0$ the following relation can be derived from Green's third formula:

$$\mathcal{K}_1(T_1^- u_1) - \mathcal{M}_1(u_1^-) = 0 \quad \text{on} \quad S. \qquad (54)$$

LEMMA 3. The operators

$$\mathcal{K}_1 : H^{-\frac{1}{2}}(S) \to H^{-\frac{1}{2}}(S), \quad \mathcal{M}_1 : H^{\frac{1}{2}}(S) \to H^{-\frac{1}{2}}(S), \qquad (55)$$

are invertible.

From (54) and Lemma3 the following Steklov-Poincaré relation for arbitrary $u_1 \in H^{1,\,0}_{loc}(\Omega^-, A_1) \cap Z(\Omega^-)$ follows

$$T_1^- u_1 = \mathcal{K}_1^{-1} \mathcal{M}_1\, u_1^- \quad \text{on} \quad S, \qquad (56)$$

where $\mathcal{K}_1^{-1}$ is the inverse to the operator $\mathcal{K}_1$.

Equivalent reduction to the system of integral equations

Let us set

$$\varphi_1 := u_1^-, \quad \varphi_2 := u_2^+, \quad \psi_1 := T_1^- u_1, \quad \psi_2 := T_2^+ u_2. \tag{57}$$

If a pair (u_1, u_2) solves the transmission problem, then in view of the notation (57), the relations obtained above and the transmission conditions can be rewritten as

$$\beta\, u_2 + \mathcal{N}_2\, u_2 - V_2(\psi_2) + W_2(\varphi_2) = \mathcal{P}_2(f) \quad \text{in} \quad \Omega^+, \tag{58}$$

$$\mathcal{N}_2^+\, u_2 - \mathcal{V}_2\psi_2 + (\beta - \mu + \mathcal{W}_2)\varphi_2 = \mathcal{P}_2^+(f) \quad \text{on} \quad S, \tag{59}$$

$$u_1 + V_1(\psi_1) - W_1(\varphi_1) = 0 \quad \text{in} \quad \Omega^-, \tag{60}$$

$$\mathcal{K}_1\psi_1 - \mathcal{M}_1\varphi_1 = 0 \quad \text{on} \quad S, \tag{61}$$

$$\varphi_2 - \varphi_1 = \varphi_0 \quad \text{on} \quad S, \tag{62}$$

$$\psi_2 - \psi_1 = \psi_0 \quad \text{on} \quad S. \tag{63}$$

Rewrite this LBDIE system in the following equivalent form

$$\beta\, u_2 + \mathcal{N}_2\, u_2 - V_2(\psi_2) + W_2(\varphi_2) = \mathcal{P}_2\big(f\big) \quad \text{in} \quad \Omega^+, \tag{64}$$

$$\mathcal{N}_2^+\, u_2 - \mathcal{V}_2\psi_2 + (\beta - \mu + \mathcal{W}_2)\varphi_2 = \mathcal{P}_2^+\big(f\big) \quad \text{on} \quad S, \tag{65}$$

$$\mathcal{K}_1\psi_2 - \mathcal{M}_1\varphi_2 = \mathcal{K}_1\psi_0 - \mathcal{M}_1\varphi_0 \quad \text{on} \quad S, \tag{66}$$

$$\varphi_1 = \varphi_2 - \varphi_0 \quad \text{on} \quad S, \tag{67}$$

$$\psi_1 = \psi_2 - \psi_0 \quad \text{on} \quad S, \tag{68}$$

$$u_1 + V_1(\psi_1) - W_1(\varphi_1) = 0 \quad \text{in} \quad \Omega^-. \tag{69}$$

Let us consider these relations as a system of equations with respect to unknowns

$$(u_2, \psi_2, \varphi_2, \psi_1, \varphi_1, u_1) \in \mathbb{H}\,, \tag{70}$$

where

$$\mathbb{H} := H^{1,0}(\Omega^+, A_2) \times H^{-\frac{1}{2}}(S) \times H^{\frac{1}{2}}(S)\times$$
$$\times H^{-\frac{1}{2}}(S) \times H^{\frac{1}{2}}(S) \times \big(H^{1,0}_{loc}(\Omega^-, A_1) \cap Z(\Omega^-)\big). \tag{71}$$

EQUIVALENCE THEOREM

The transmission problem (15)-(20) and the system of integral equations (64)-(69) are equivalent in the following sense:

(i) If a pair $(u_2, u_1) \in H^{1,0}(\Omega^+, A_2) \times \big(H^{1,0}_{loc}(\Omega^-, A_1) \cap Z(\Omega^-)\big)$ solves the transmission problem (15)-(20), then the vector

$$(u_2, \psi_2, \varphi_2, \psi_1, \varphi_1, u_1) \in \mathbb{H}$$

with $\psi_2, \varphi_2, \psi_1, \varphi_1$, defined by the equalities

$$\varphi_1 := u_1^-, \quad \varphi_2 := u_2^+, \quad \psi_1 := T_1^- u_1, \quad \psi_2 := T_2^+ u_2, \qquad \textbf{(72)}$$

solves the system (64)-(69), and vice versa,

(ii) If a vector $(u_2, \psi_2, \varphi_2, \psi_1, \varphi_1, u_1) \in \mathbb{H}$ solves the system (64)-(69), then the pair

$$(u_2, u_1) \in H^{1,0}(\Omega^+, A_2) \times \big(H^{1,0}_{loc}(\Omega^-, A_1) \cap Z(\Omega^-)\big)$$

solves the transmission problem (15)-(20) and the relations (72) hold true.

Proof. **(i)** The first part of the theorem is trivial.

(ii) Let a vector $(u_2, \psi_2, \varphi_2, \psi_1, \varphi_1, u_1) \in \mathbb{H}$ solve the system (64)-(69). Taking the trace of (64) on S and comparing with (65) lead to the equation

$$u_2^+ = \varphi_2 \quad \text{on} \quad S. \tag{73}$$

Further, since $u_2 \in H^{1,0}(\Omega_2, A_2)$ we can write Green's third identity (22) which in view of (73) can be rewritten as

$$\begin{aligned} \beta(y)\, u_2(y) + \mathcal{N}_2\, u_2(y) - V_2(T_2^+ u_2)(y) + W_2(\varphi_2)(y) = \\ = \mathcal{P}_2\big(A_2 u_2\big)(y), \quad y \in \Omega_2. \end{aligned} \tag{74}$$

From (64) and (74) it follows that

$$V_2(T_2^+ u_2 - \psi_2) + \mathcal{P}_2\big(A_2 u_2 - f_2\big) = 0 \quad \text{in} \quad \Omega_2, \tag{75}$$

implying

$$A_2 u_2 = f_2 \quad \text{in} \quad \Omega_2, \qquad T_2^+ u_2 = \psi_2 \quad \text{on} \quad S. \tag{76}$$

Further, from equation (69) it follows that

$$A_1 u_1 = 0 \ \text{ in } \ \Omega_1, \qquad \mathcal{K}_1 \psi_1 - \mathcal{M}_1 \varphi_1 = 0 \ \text{ on } \ S. \tag{77}$$

Now, let us consider the function

$$v := V_1(\psi_1) - W_1(\varphi_1) \quad \text{in} \quad \Omega_2 = \Omega^+. \tag{78}$$

It can be shown that

$$A_1 v = 0 \ \text{ in } \ \Omega_1, \tag{79}$$

$$T_1^+ v - i\,\alpha\, v^+ = 0 \ \text{ on } \ S, \quad \alpha \in \mathbb{R} \setminus \{0\}, \tag{80}$$

which implies that v vanishes identically in $\Omega_2 = \Omega^+$, since the Robin problem (79)-(80) possess only the trivial solution. Therefore

$$u_1^- = u_1^- + v^+ = \varphi_1, \qquad T_1^- u_1 = T_1^- u_1 + T_1^+ v = \psi_1. \tag{81}$$

which completes the proof. ■

Uniqueness Theorem. **The homogeneous transmission problem possesses only the trivial solution.**

Proof. Let $B(R)$ be a ball centered at the origin and radius R, such that $\overline{\Omega^+} \subset B(R)$. Let a pair (u_1, u_2) be a solution to the homogeneous transmission problem. Write Green's formulas for the domains Ω^+ and $\Omega_R^- = \Omega^- \cap B(R)$

$$\int_{\Omega^+} [a_{kj}^{(2)}(x)\partial_j u_2(x)\overline{\partial_k u_2(x)} - \omega^2 \kappa(x)|u_2(x)|^2] dx = \langle T_2^+ u_2 \, , \, \overline{u_2^+} \rangle_S ,$$

$$\int_{\Omega_R^-} [a_{kj}^{(1)} \, \partial_j u_1(x) \, \overline{\partial_k u_1(x)} - \omega^2 |u_1(x)|^2] \, dx = -\langle T_1^- u_1 \, , \, \overline{u_1^-} \rangle_S + $$

$$+ \langle T_1 u_1 \, , \, \overline{u_1} \rangle_{\Sigma(R)} .$$

Since the matrix $\mathbf{a} = [a_{kj}]_{3\times 3}$ is symmetric and positive definite, in view of the homogeneous transmission conditions we get

$$\mathrm{Im} \left\{ \int_{\Sigma_R} T_1(x, \partial) u_1(x) \, \overline{u_1(x)} \, d\Sigma_R \right\} = 0. \qquad \textbf{(82)}$$

Since $u_1 \in Z(\Omega^-)$ we get

$$\overline{u_1(x)}T_1(x,\partial)u_1(x) = i\,\omega\,(\mathrm{b}\eta\cdot\eta)^{-\frac{1}{2}}\,|u_1(x)|^2 + \mathcal{O}(|x|^{-3}). \quad \textbf{(83)}$$

Due to positive definiteness of the matrix $\mathrm{b} := [\mathrm{a}^{(1)}]^{-1}$, there are positive constants δ_0 and δ_1 such that for all $\eta \in \Sigma_1$

$$0 < \delta_0 \leqslant (\mathrm{b}\eta\cdot\eta)^{-\frac{1}{2}} \leqslant \delta_1 < \infty\,. \quad \textbf{(84)}$$

Therefore

$$\textbf{(82)} \quad \Rightarrow \quad \lim_{R\to+\infty}\int_{\Sigma_R} |u_1(x)|^2\,d\Sigma_R = 0 \quad \textbf{(85)}$$

and by Rellich-Vekua Lemma $u_1 = 0$ in Ω^-.

Consequently, the function u_2 solves the homogeneous Cauchy problem in Ω^+ for the elliptic partial differential equation

$$A_2(x,\partial)u_2 = 0 \text{ in } \Omega^+,$$
$$\{u_2\}^+ = 0, \quad \{T_2 u_2\}^+ = 0 \text{ on } S,$$

with variable coefficients $a_{kj}^{(2)}$ and κ_2 with $a_{kj}^{(2)}, \kappa_2 \in C^2(\overline{\Omega}_2)$.
By the interior and boundary regularity properties of solutions to elliptic equations we have that $u_2 \in C^2(\overline{\Omega^+})$ and therefore $u_2 = 0$ in Ω^+ due to the well known uniqueness theorem for the Cauchy problem (E.M. Landis - 1956, A.P.Calderon - 1958). ■

Consequently, the function u_2 solves the homogeneous Cauchy problem in Ω^+ for the elliptic partial differential equation

$$A_2(x,\partial)u_2 = 0 \ \text{ in } \ \Omega^+,$$
$$\{u_2\}^+ = 0, \quad \{T_2 u_2\}^+ = 0 \ \text{ on } \ S,$$

with variable coefficients $a_{kj}^{(2)}$ and κ_2 with $a_{kj}^{(2)}, \kappa_2 \in C^2(\overline{\Omega}_2)$.
By the interior and boundary regularity properties of solutions to elliptic equations we have that $u_2 \in C^2(\overline{\Omega^+})$ and therefore $u_2 = 0$ in Ω^+ due to the well known uniqueness theorem for the Cauchy problem (E.M. Landis - 1956, A.P.Calderon - 1958). ■

COROLLARY 4. The LBDIE system (64)-(69) possesses at most one solution in the space $\mathbb{H}$.

Consequently, the function u_2 solves the homogeneous Cauchy problem in Ω^+ for the elliptic partial differential equation

$$A_2(x, \partial)u_2 = 0 \;\text{ in }\; \Omega^+,$$
$$\{u_2\}^+ = 0, \quad \{T_2 u_2\}^+ = 0 \;\text{ on }\; S,$$

with variable coefficients $a_{kj}^{(2)}$ and κ_2 with $a_{kj}^{(2)}, \kappa_2 \in C^2(\overline{\Omega}_2)$.
By the interior and boundary regularity properties of solutions to elliptic equations we have that $u_2 \in C^2(\overline{\Omega^+})$ and therefore $u_2 = 0$ in Ω^+ due to the well known uniqueness theorem for the Cauchy problem (E.M. Landis - 1956, A.P.Calderon - 1958). ■

COROLLARY 4. The LBDIE system (64)-(69) possesses at most one solution in the space $\mathbb{H}$.

REMARK 5: Due to the recent results, the uniqueness theorem for the Cauchy problem for a scalar elliptic operator holds if variable coefficients are Lipschitz continuous and Ω^+ is a Dini domain (X.X.Tao and S.Y.Zhang - 2007).

Investigation of the LBDIE system

$$(\beta I + \mathrm{N}_2)\, E_0\, u_2 - V_2(\psi_2) + W_2(\varphi_2) = \mathcal{P}_2\big(f_2\big) \quad \text{in} \quad \Omega^+, \qquad (86)$$

$$\mathrm{N}_2^+\, E_0\, u_2 - \mathcal{V}_2\psi_2 + (\beta - \mu + \mathcal{W}_2)\varphi_2 = \mathcal{P}_2^+\big(f_2\big) \quad \text{on} \quad S, \qquad (87)$$

$$\mathcal{K}_1\psi_2 - \mathcal{M}_1\varphi_2 = \mathcal{K}_1\psi_0 - \mathcal{M}_1\varphi_0 \quad \text{on} \quad S, \qquad (88)$$

$$\varphi_1 = \varphi_2 - \varphi_0 \quad \text{on} \quad S, \qquad (89)$$

$$\psi_1 = \psi_2 - \psi_0 \quad \text{on} \quad S, \qquad (90)$$

$$u_1 + V_1(\psi_1) - W_1(\varphi_1) = 0 \quad \text{in} \quad \Omega^-. \qquad (91)$$

where E_0 denotes the extension operator by zero from Ω^+ onto Ω^-.

We need to investigate the matrix operator generated by the left hand side expressions in the first three equations. If the unknowns u_2, ψ_2, and φ_2 are found from the first three equations, then the unknowns ψ_1, φ_1, u_1 can be defined explicitly from the last three equations.

Let us rewrite the first three equations of the LBDIE system (86)-(91) in matrix form

$$\mathrm{M}\, U^{(2)} = F, \tag{92}$$

$$\mathrm{M} := \begin{bmatrix} r_{\Omega^+}(\beta\, I + \mathrm{N}_2)E_0 & -r_{\Omega^+} V_2 & r_{\Omega^+} W_2 \\ \mathrm{A}_2^+ E_0 & -\mathcal{V}_2 & (\beta - \mu)I + \mathcal{W}_2 \\ 0 & \mathcal{K}_1 & -\mathcal{M}_1 \end{bmatrix} \tag{93}$$

$$U^{(2)} := (u_2, \psi_2, \varphi_2)^\top, \qquad F := \big(F_1, F_2, F_3\big)^\top, \tag{94}$$

Applying the mapping properties of the layer potentials and pseudodifferential operators with rational type symbols we deduce that the following operator is continuous

$$\mathrm{M} \;:\; \mathbb{X}^1 \to \mathbb{Y}^1, \tag{95}$$

$$\mathbb{X}^1 := H^1(\Omega^+) \times H^{-\frac{1}{2}}(S) \times H^{\frac{1}{2}}(S), \tag{96}$$

$$\mathbb{Y}^1 := H^1(\Omega^+) \times H^{\frac{1}{2}}(S) \times H^{-\frac{1}{2}}(S). \tag{97}$$

THEOREM 4. **The operator**

$$\mathbf{M} \ : \ \mathbb{X}^{\mathbf{1}} \rightarrow \mathbb{Y}^{\mathbf{1}}$$

is invertible.

THEOREM 5. The operator

$$\mathbf{M} \;:\; \mathbb{X}^{1} \to \mathbb{Y}^{1}$$

is invertible.

Proof. Invertibility of the operator M can be shown with the help of the Vishik-Eskin theory. The proof is performed into several steps.

Step 1. The operator

$$\mathbf{M}_{11} = r_{\Omega^+}(\beta\, I + \mathbf{N}_2)E_0 \;:\; H^1(\Omega^+) \to H^1(\Omega^+)$$

is Fredholm with zero index.
The proof is based on the fact that the principal homogeneous symbol

$$\mathfrak{S}_0(\mathbf{D}_{11}; y, \xi) = \frac{a_{kl}^{(2)}(y)\xi_k\xi_l}{|\xi|^2} > 0, \quad y \in \overline{\Omega^+}, \quad \xi \in \mathbb{R}^3 \setminus \{0\},$$

is positive, rational, homogeneous of order $\mathbf{0}$ in ξ.

Step 2. **The operator $\mathbf{M}$ is Fredholm with zero index.**

(i) **Fredholm property follows from the fact that for the operator $\mathbf{M}$ the generalized Šapiro-Lopatinskiĭ condition holds which is verified by direct calculations of specific Cauchy type integrals.**

(ii) **The zero index is established by showing that $\mathbf{M}$ is homotopic to an invertible operator.**

Step 2. **The operator $\mathbf{M}$ is Fredholm with zero index.**

(i) **Fredholm property follows from the fact that for the operator $\mathbf{M}$ the generalized Šapiro-Lopatinskiĭ condition holds which is verified by direct calculations of specific Cauchy type integrals.**

(ii) **The zero index is established by showing that $\mathbf{M}$ is homotopic to an invertible operator.**

Step 3. **The null space of the operator $\mathbf{M}$ is trivial.**
This is shown with the help of the Equivalence and Uniqueness Theorems.

Consequently, the operator $\mathbf{M}$ is invertible.

COROLLARY 6. The operator

$$\mathrm{M} \; : \; H^{1,0}(\Omega^+, A_2) \times H^{-\frac{1}{2}}(S) \times H^{\frac{1}{2}}(S) \rightarrow$$
$$\rightarrow H^{1,0}(\Omega^+, \Delta) \times H^{\frac{1}{2}}(S) \times H^{-\frac{1}{2}}(S) \quad \mathbf{(98)}$$

is invertible.

COROLLARY 7. Let a cut-off function $\chi \in X_+^\infty$ and $r \geq 0$.
Then the operator

$$\mathrm{M} \; : \; H^{r+1}(\Omega^+) \times H^{r-\frac{1}{2}}(S) \times H^{r+\frac{1}{2}}(S) \rightarrow$$
$$\rightarrow H^{r+1}(\Omega^+) \times H^{r+\frac{1}{2}}(S) \times H^{r-\frac{1}{2}}(S) \quad \mathbf{(99)}$$

is invertible.

From the above results it follows that the LBDIE system

$$(\beta\, I + \mathrm{N}_2)\, E_0\, u_2 - V_2(\psi_2) + W_2(\varphi_2) = h_1 \quad \text{in} \quad \Omega^+, \tag{100}$$

$$\mathrm{A}_2^+\, E_0\, u_2 - \mathcal{V}_2\psi_2 + (\beta - \mu + \mathcal{W}_2)\varphi_2 = h_2 \quad \text{on} \quad S, \tag{101}$$

$$\mathcal{K}_1\psi_2 - \mathcal{M}_1\varphi_2 = h_3 \quad \text{on} \quad S, \tag{102}$$

$$\psi_2 - \psi_1 = h_4 \quad \text{on} \quad S, \tag{103}$$

$$\varphi_2 - \varphi_1 = h_5 \quad \text{on} \quad S, \tag{104}$$

$$u_1 + V_1(\psi_1) - W_1(\varphi_1) = h_6 \quad \text{in} \quad \Omega^-. \tag{105}$$

is uniquely solvable in the space

$$\mathbb{X} := H^1(\Omega^+) \times H^{-\frac{1}{2}}(S) \times H^{\frac{1}{2}}(S) \times H^{-\frac{1}{2}}(S)\times$$
$$\times H^{\frac{1}{2}}(S) \times \big(H^1_{loc}(\Omega^-) \cap Z(\Omega^-)\big)$$

for arbitrary right hand side data $(h_1, \cdots, h_6) \in \mathbb{Y}$ with

$$\mathbb{Y} := H^1(\Omega^+) \times H^{\frac{1}{2}}(S) \times H^{-\frac{1}{2}}(S) \times H^{-\frac{1}{2}}(S)\times$$
$$\times H^{\frac{1}{2}}(S) \times H^1_{comp}(\Omega^-).$$

THE PROBLEMS OF ACOUSTIC SCATTERING BY INHOMOGENEOUS ANISOTROPIC OBSTACLES

are uniquely solvable.

THANK YOU!