References - nonlinear Liouville's theorems


4. Nonlinear Liouville type theorems.

Consider semi-linear equation

\( \star \) \hspace{1cm} -\Delta u = u^p \quad \text{in} \quad \Omega = \mathbb{R}^n \setminus \overline{B} , \quad n \geq 3 , \quad p \in \mathbb{R} \\

\( 0 \leq u \in \text{H}^1_{00} (\Omega) \) is a nonnegative weak super solution if

\[ \int_{\Omega} u \varphi \geq \int_{\Omega} u^p \varphi , \quad \forall \varphi \in \text{C}_c^\infty (\Omega) , \quad \varphi \geq 0. \]

If \( u \geq 0 \) is a supersolution \( \Rightarrow \) \hspace{1cm} -\Delta u \geq 0 \hspace{1cm} \text{in} \hspace{1cm} \Omega \\

\( \Rightarrow \) \hspace{1cm} u > 0 \text{ \ a.e. in } \Omega . \)
Theorem (Serrin \(= 46\))

\(\star\) has a positive supersolution \(\iff p > \frac{n}{n-2}\)

Difficult part is nonexistence. We consider separately \(p > 1\) and \(p < 1\).

If \(p = 1\) \(\star\) is linear: \(-\Delta u = u\) in \(\Omega\)

Then \(E(u) = \int_{\Omega}|\nabla u|^2 - \int_{\Omega}|u|^2 < 0\) for some \(u \in C_0\)

\(\implies \star\) has no positive supersolution by AAP
Case $p \geq 1$.

Assume $\exists u > 0$, $-\Delta u \geq u^p$ in $\Omega$.

Then $-\Delta u \geq 0$ in $\Omega \Rightarrow u \geq c_1 |x|^{-(n-2)(p-1)}$ in $B_2^c$ (1

(|x|^{-(n-2)}$ is a small solution to $-\Delta$!

$)$

$\Rightarrow -\Delta u - V_1(x) u \geq 0$ in $\Omega$ - linearisation

By (1), $V_1(x) = u^{p-1} \geq c_1 |x|^{-(n-2)(p-1)}$

$\Rightarrow -\Delta u - c_1 |x|^{-(n-2)(p-1)} u \geq 0$ in $\Omega$

$\Rightarrow 1 < p < \frac{n}{n-2}$ $\Rightarrow -(n-2)(p-1) > -2$

$\Rightarrow -2 + \varepsilon$
Consider $E(u) = \int_{\Omega} |Du|^2 - c_1 \int_{\Omega} \frac{u^2}{|x|^2 - \varepsilon}$

Take $u > 0$, $u \in C^\infty_0 (A_{1,2})$

Set $u_R(x) = u \left( \frac{x}{R} \right)$. Then

$E(u_R) = R^{n-2} \int_{\Omega} |D_u|^2 - c_1 R^{n-2+\varepsilon} \int_{\Omega} \frac{u^2}{|x|^2 - \varepsilon}$

$\Rightarrow$ as $R \to \infty$

$\Rightarrow$ (**) has no positive supersolution.
\[
1 < p < \frac{N}{N-2} \implies -\Delta u \geq u^p \text{ has no pos. supersol in } \mathbb{S}^2
\]

Consider critical case \[ p = \frac{N}{N-2} \quad \left( - \frac{(N-2)(p-1)}{2} = -2 \right) \]

Then \[-\Delta u - \frac{c_1}{\|x\|^2} u \geq 0 \text{ in } \mathbb{S}^2 \quad (\ast\ast)\]

For some \( c_1 > 0 \).

If \( c_1 > c_H = \left(\frac{N-2}{2}\right)^2 \) \[ (\ast\ast) \text{ has no pos. supersol. by AAP.} \]

Assume \( c_1 \) is small!
Then \(-\Delta u - \frac{c_1}{|x|^2} u \geq 0\) in \(B_{\frac{1}{2}}\), \(0 < c_1 < C_H\).

\[ \Rightarrow u \geq c_2 |x|^{2-} \quad \text{--- small solution of } -\Delta - \frac{c_1}{|x|^2} \]

\(2-\) is the smallest root of \(-2(t+n-2) = c_1\).

Note that \(-(n-2) < 2- < -\frac{n-2}{2}!\) Then

\[ -\Delta u - V_2(x) u \geq 0 \text{ in } \Sigma \]

\[ V_2(x) = u^{p-1} \geq c_3 |x|^{2-(p-1)} = c_3 |x|^{-2+\varepsilon} \]

By Lemma 1 --- no pos supersoln!
Case $p < 1$. \(-\Delta u - u^{p-1} u \geq 0\) 
\[ p-1 < 0 \implies \text{upper bound on } u \text{ is needed!} \]

But upper bound on superharmonics \(-\Delta u \geq 0\) from large solution is not pointwise!

\[-\Delta u \geq 0 \text{ in } B^c \implies \liminf_{|x| \to 0} u \leq +\infty !\]
Lemma 2. \(-\Delta u \geq u^p\) in \(B_1\), \(p < 1\)

\[ \Rightarrow u \geq c|x|^\frac{2}{1-p} \]

\(\triangleleft\) See [3, Lemma 6.17] — uses AAP + weak Harnack

But by Phragmen-Lindelöf,

\(-\Delta u \geq 0\) in \(B_1\) \(\Rightarrow\) \(\lim_{|x| \to \infty} u < +\infty\)

— incompatible with \(u \geq c|x|^\frac{2}{p-1} \geq 0\), \(p < 1\)!
We proved nonexistence \( \nexists \ p \leq \frac{N}{N-2} \):

\[
p = 1 \Rightarrow 1 < p < \frac{N}{N-2} \quad p = \frac{N}{N-2} \quad p < 1.
\]

Lemma. Assume \( p > \frac{N}{p-2} \). Then 

\[c|x|^{-\frac{2}{p-1}}\] is a solution of \(-\Delta u = u^p\) in \(\mathbb{R}^N \setminus \{0\}\).

So some explicit \( c = c(N, p) \).

Existence proved. 

\[-(N-2) < -\frac{2}{p-1}\]

Remark: \( \exists \ \text{sol.} \ u > 0 \), \( u = |x|^{-\frac{(N-2)}{2}} \) as \( |x| \to \infty \).
Exercise: \(-\Delta u + \frac{c}{|x|^2} u \geq u^p\) in \(B_1^c, c > 0\)

has a positive supersolution \(p \in \left[1 - \frac{2}{d+1}, 1 - \frac{2}{d-1}\right]\),

where \(1_- < 1_+\) are roots of \(-2(2+n-2)+c=0\).

Hint: \(p > 1\) — the same

\(p < 1\) — the same \(\odot\) Lemma 2 remains valid.
\[-\Delta V + \frac{c}{|x|^2} V = 0 \text{ in } B_1^c, \quad c > 0\]

\[V_0 = |x|^{d+1} - 1\]

- Large solution

\[V_1 = |x|^d\]
\[-\Delta u + u^p = 0\]


\[-\Delta u - \frac{c}{|x|^l} u + \frac{\theta}{|x|^2} u \text{ in } \mathbb{R}^n\]