Mathematical Analysis of problems in Complex Media Electromagnetics

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Mini-course (handwritten) Notes
Lectures 2 – 5
\( \Omega \): open, bounded, connected and simply connected in \( \mathbb{R}^2 \)

\( \partial \Omega \): bounded, connected, \( C^{1,1} \) (graph of an everywhere differentiable function with locally Lipschitz continuous gradient)

\( n(x) \): the unit outward normal at \( x \in \partial \Omega \)

**THE INTERIOR PROBLEM**

\[
\begin{align*}
\text{curl} \mathbf{E}(x) &= \mathbf{B}(x) \gamma^2(x) \mathbf{E}(x) + i \omega \mu(x) \left( \frac{\gamma(x)}{k(x)} \right)^2 \mathbf{H}(x) , \quad x \in \Omega \\
\text{curl} \mathbf{H}(x) &= \mathbf{B}(x) \gamma^2(x) \mathbf{H}(x) - i \omega \epsilon(x) \left( \frac{\gamma(x)}{k(x)} \right)^2 \mathbf{E}(x) \\
n(x) \times \mathbf{E}(x) &= f(x) , \quad x \in \partial \Omega
\end{align*}
\]

- \( \omega > 0 \): angular frequency
- \( k^2(x) := \omega^2 \epsilon(x) \mu(x) \), \( \gamma^2(x) := \frac{k^2(x)}{1 - \beta^2(x) k^2(x)} \)
- \( \epsilon, \mu, \beta : \overline{\Omega} \to \mathbb{R} \) positive, \( C^2 \)-functions
- \( \frac{1 - \omega^2 \epsilon(x) \mu(x) \beta^2(x)}{\mu(x)} > 0 \)

If \( f = 0 \) the problem is homogeneous; as long as the other parameters are fixed, the homogeneous problem will have nontrivial solutions for specific values of \( \omega \) (eigenvalues/resonant frequencies of \( \Omega \) /cavity problem).

Assume that we are away from such an eigenvalue.

Then, we will prove that the interior problem is well-posed.
Before we state the corresponding theorem, we will discuss FUNCTION SPACES

\[ H^m(\Omega) = W^{m,2}(\Omega), \quad m \in \mathbb{N}_0, \quad (W^{0,2} = L^2) \]

\[ H^m_0(\Omega) = \text{closure of } C^\infty_0(\Omega) \text{ in the } H^m(\Omega) \text{ norm} \]

\[ H^s(\Omega) = \left\{ u \in L^2(\Omega): \frac{|u(x) - u(y)|}{|x-y|^{s+\frac{3}{2}}} \in L^2(\Omega \times \Omega) \right\} \quad (0 < s < 1) \]

\[ H^s(\Omega) = \left\{ u \in H^{[s]}(\Omega): D^\alpha u \in H^{s-[\alpha]}(\Omega), \quad \forall \alpha \in \mathbb{N}_0^N, \quad |\alpha| = [s] \right\} \]

(\[ [s] \]: the biggest integer that is less or equal than s)

\[ \| u \|_{H^s(\Omega)} = \left( \| u \|_{H^{[s]}(\Omega)}^2 + \sum_{|\alpha| = [s]} \int_{\Omega \times \Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{3+2(s-|\alpha|)}} \, dx \, dy \right)^{1/2} \]

\[ H^s_0(\Omega) = \text{closure of } C^\infty_0 \text{ in the } H^s(\Omega) \text{ norm} \]

\[ H^{-s}(\Omega) = \text{the dual space of } H^s_0(\Omega) \]

With our assumptions on \( \Omega \) and \( \partial \Omega \), there exists a unique linear continuous map (the "trace" of \( u \) on \( \partial \Omega \)) \( \chi_0: H^1(\Omega) \to L^2(\partial \Omega) \): for any \( u \in H^1(\Omega) \cap C(\overline{\Omega}) \)

\[ \chi_0(u) = u \big|_{\partial \Omega} \]

\[ H^{\frac{1}{2}}(\partial \Omega) := \chi_0(H^1(\Omega)) \]

\[ H^{-\frac{1}{2}}(\partial \Omega) := \text{the dual space of } H^{\frac{1}{2}}(\partial \Omega) \]
To define $H^s$ on $\Gamma$, assume first that $\Gamma$ is of the special form
\[ \Gamma_0 = \{(y, \psi(y)) : y \in \mathbb{R}^2\} \]
$\psi : \mathbb{R}^2 \to \mathbb{R}$ a $C^{k-1,1}$ function ($k \in \mathbb{N}$)

Then set
\[ u_\psi(y) = u(y, \psi(y)), y \in \mathbb{R}^2 \]

and then
\[ H^s(\Gamma_0) := \{ u \in L^2(\Gamma_0) : u_\psi \in H^s(\mathbb{R}^2) \}, 0 \leq s \leq k. \]

This is a Hilbert space, equipped with the inner product
\[ (u, v)_{H^s(\Gamma_0)} := (u_\psi, v_\psi)_{H^s(\mathbb{R}^2)}. \]

Further, $H^{-s}(\Gamma_0)$ is the dual space of $H^s(\Gamma_0)$.

When $\Gamma$ is a general $C^{k-1,1}$ surface, we consider a finite cover with each $\Gamma_j$ transformed by rotation and translation to the form of $\Gamma_0$, and consider a partition of unity associated to this cover. Then
\[ u \in H^s(\Gamma) \iff \Theta_j u \in H^s(\Gamma_j) \]

and
\[ \|u\|_{H^s(\Gamma)} = \sum_j \|\Theta_j u\|_{H^s(\Gamma_j)} \]

which is proved to be independent of the choice of the partition of unity.

$H^{-s}(\Gamma)$ is the dual space of $H^s(\Gamma)$.

Note that
\[ H^s(\partial \Omega) \] can be seen as the space of traces of $H^{s+\frac{1}{2}}(\Omega)$.

Fractional index Sobolev spaces also arise as interpolation spaces.
Spaces of tangential fields

\[ v : \partial \Omega \to \mathbb{R}^3 \quad \Rightarrow \quad v\big|_{\partial \Omega} = (n \cdot v)\big|_{\partial \Omega} n + (n \times v)\big|_{\partial \Omega} \times n \]

**Normal Trace Operator**

\[ \gamma_n : v \mapsto n \cdot v\big|_{\partial \Omega} \]

**Tangential Trace Operator**

\[ \gamma_T : v \mapsto n \times v\big|_{\partial \Omega} \]

**Tangential Components Trace Operator**

\[ \Pi : v \mapsto (n \times v)\big|_{\partial \Omega} \times n \]

- \[ TL^2(\partial \Omega) = \{ u \in (L^2(\partial \Omega))^3 : \gamma_n(u) = 0 \} \]
- \[ TH^6(\partial \Omega) = \{ u \in (H^6(\partial \Omega))^3 : \gamma_n(u) = 0 \} \]

**Tangential Differential Operators**

- **Vector Valued acting on Scalar Fields**
  - \( \text{Grad} q := \Pi_{\partial \Omega} (\text{grad} \Phi) \)
    
    \[ q : \partial \Omega \to \mathbb{R} \]
    
    \( \Phi : \overline{\Omega} \to \mathbb{R} : \Phi|_{\partial \Omega} = q \]  
    
    (extension)

  - \( \overrightarrow{\text{Curl}} q := -\gamma_T(\text{Grad} q) \)

- **Scalar acting on (tangential) Vector Fields**
  - \( \text{Curl} v := \gamma_n(\text{curl} v) \)
    
    \[ v : \partial \Omega \to \mathbb{R}^3 \]
    
    \( v : \overline{\Omega} \to \mathbb{R}^3 : v|_{\partial \Omega} = v \]  
    
    (extension)

  - \( \langle \text{Div} v, \overrightarrow{\Phi} \rangle_{L^2(\partial \Omega)} := -\langle v, \text{Grad} \overrightarrow{\Phi} \rangle_{L^2(\partial \Omega)} \)  
    
    \( L^2(\partial \Omega) \times (L^2(\partial \Omega))^3 \), \( \forall \overrightarrow{\Phi} \in C_c^\infty(\partial \Omega) \)

**Laplace–Beltrami Operator**

\[ \Delta_{\partial \Omega} q := \text{Div} \text{Grad} q = -\text{Curl} \overrightarrow{\text{Curl}} q \]
We also need the following Hilbert space

\[ H^{s}(\text{div}, \Omega) = \left\{ \mathbf{u} \in \mathbf{H}^{s}(\Omega) : \text{Div} \mathbf{u} \in H^{s}(\Omega) \right\} \]

endowed with the norm

\[ \| \mathbf{u} \|_{H^{s}(\text{div}, \Omega)}^2 = \| \mathbf{u} \|_{\mathbf{H}^{s}(\Omega)}^2 + \| \text{Div} \mathbf{u} \|_{H^{s}(\Omega)}^2 \]

For \( s = -\frac{1}{2} \) the space \( H^{-1/2}(\text{div}, \Omega) \) is the natural energy space which occurs in E/M potential theory.

**Spaces for Electromagnetics**

- \( H(\text{curl}, \Omega) = \left\{ \mathbf{u} \in (L^2(\Omega))^3 : \text{curl} \mathbf{u} \in L^2(\Omega) \right\} \)

  \[ \| \mathbf{u} \|_{H(\text{curl}, \Omega)}^2 = \| \mathbf{u} \|_{(L^2(\Omega))^3}^2 + \| \text{curl} \mathbf{u} \|_{L^2(\Omega)}^2 \]

  \( (C_0^1(\overline{\Omega}))^3 \) and \( (C_0^\infty(\overline{\Omega}))^3 \) are dense in \( H(\text{curl}, \Omega) \).

  This is the energy space in E/M. Further,

  - \( H_0(\text{curl}, \Omega) = \left\{ \mathbf{u} \in H(\text{curl}, \Omega) : \mathbf{n} \times \mathbf{u} \big|_{\partial \Omega} = \mathbf{0} \right\} \) \( \text{for } \Omega \text{ bounded} \)

  - \( H(\text{div}, \Omega) = \left\{ \mathbf{u} \in (L^2(\Omega))^3 : \text{div} \mathbf{u} \in L^2(\Omega) \right\} \)

    \[ \| \mathbf{u} \|_{H(\text{div}, \Omega)}^2 = \| \mathbf{u} \|_{(L^2(\Omega))^3}^2 + \| \text{div} \mathbf{u} \|_{L^2(\Omega)}^2 \]

- \( H_0(\text{div}, \Omega) = \left\{ \mathbf{u} \in H(\text{div}, \Omega) : \mathbf{n} \cdot \mathbf{u} \big|_{\partial \Omega} = \mathbf{0} \right\} \) \( \text{for } \Omega \text{ bounded} \)

- \( H(\text{div}_0, \Omega) = \left\{ \mathbf{u} \in (L^2(\Omega))^3 : \text{div} \mathbf{u} = 0 \right\} \)

  (and \( H(\text{curl}_0, \Omega), H_0(\text{div}_0, \Omega), H_0(\text{curl}_0, \Omega), H(\text{curl}, \text{div}, \Omega), \ldots \))

The corresponding spaces for exterior problems, in \( \Omega_{\text{ext}} = \mathbb{R}^3 \setminus \Omega \), are indicated by \( \mathbf{X}_{\text{loc}} \), but we won't deal with them.
THEOREM

Assume that \( w \) is not an eigenvalue of

1. \( \text{curl} E = \beta \xi^2 E + iw\psi \left( \frac{\xi}{k} \right)^2 H \)
2. \( \text{curl} H = \beta \xi^3 H - i\omega \xi \left( \frac{\xi}{k} \right)^2 E \), \text{ in } \Omega.
3. \( n \times E = 0 \), \text{ on } \partial\Omega.

Then the interior problem (1), (2)

4. \( n \times E = f \), \text{ on } \partial\Omega,

has a unique solution in \( H(\text{curl}, \Omega) \times H(\text{curl}, \Omega) \) for every \( f \in H^{-\frac{1}{2}}(\text{div}, \partial\Omega) \), under the assumptions

5. \( \varepsilon, \mu, \beta : \overline{\Omega} \to \mathbb{R} \) are positive \( C^2(\overline{\Omega}) \) functions
6. \( \frac{1 - \omega^2 \varepsilon \mu^2}{\mu} > 0 \), \text{ in } \overline{\Omega}.

and the considered assumptions on \( \Omega \) and \( \partial\Omega \).

PROOF

1. Rewrite (1), (2) in terms of \( E \) only:

\[
\text{curl} \left( \frac{1 - \omega^2 \varepsilon \mu^2}{\mu} \text{curl} E \right) = \omega^2 \left[ \text{curl} (\beta \xi^3 E) + \beta \xi \text{curl} E + \xi E \right]
\]

Denote \( \theta := \frac{1 - \omega^2 \varepsilon \mu^2}{\mu} \).

Weak formulation: take dot product of above eqn with a vector function \( \psi \), integrate over \( \Omega \) and use the Green's 2nd formula

\[
\int_{\Omega} \text{curl} \psi \cdot w \, dx = -\int_{\Omega} \text{curl} \psi \cdot \text{curl} w \, dx = \left\langle \gamma_1(v), \Pi_\partial(w) \right\rangle, \forall \psi, w \in H(\text{curl}, \Omega)
\]

The first term becomes

\[
\int_{\Omega} \text{curl}(\theta \text{curl} E) \cdot \overline{\psi} \, dx = \int_{\Omega} (\theta \text{curl} E) \cdot \text{curl} \overline{\psi} \, dx + \int_{\partial\Omega} (\theta \text{curl} E) : ((n \times \overline{\psi}) \times n) \, ds(n)
\]

If \( \psi \in H_0^1(\text{curl}, \Omega) \), then \( \int_{\partial\Omega} = 0 \),
so we get
\[\int_{\Omega} (\theta \text{curl} \mathbf{E}) \cdot \text{curl} \Phi = \omega^2 \int_{\Omega} (\mathbf{E} \mathbf{E}^T + \text{curl} (\beta \mathbf{E} \mathbf{E}) + \beta \text{curl} \mathbf{E} \mathbf{E}^T) \cdot \Phi \, dx\]

and hence we introduce the bilinear form
\[a^{(p)}(\cdot, \cdot) : H(\text{curl}, \Omega) \times H_0(\text{curl}, \Omega) \to \mathbb{C}\]
\[a^{(p)}(\mathbf{E}, \Phi) = (\theta \text{curl} \mathbf{E}, \text{curl} \Phi) - \omega^2 (\mathbf{E} \mathbf{E}^T, \Phi) - \omega^2 (\text{curl} (\beta \mathbf{E} \mathbf{E}), \Phi) - \omega^2 (\beta \text{curl} \mathbf{E}, \Phi),\]

where \((\cdot, \cdot)\) is the \((L^2(\Omega))^3\) inner product.

2. We make homogeneous the boundary condition \(n \mathbf{E} = f\).

Seek \(\mathbf{E} = \mathbf{U} + \mathbf{E}_0\), \(\mathbf{U} \in (H^1(\Omega))^3, \lambda(\mathbf{U}) = f\).

\(\mathbf{E}_0\) satisfies

- \(\text{curl}(\theta \text{curl} \mathbf{E}_0) - \omega^2 [\text{curl}(\beta \mathbf{E} \mathbf{E}_0) + \beta \text{curl} \mathbf{E}_0 + \mathbf{E} \mathbf{E}_0] = f, \text{ in } \Omega,\)

where \(f := -\text{curl}(\beta \text{curl} \mathbf{U}) + \omega^2 [\text{curl}(\beta \mathbf{E} \mathbf{U}) + \beta \text{curl} \mathbf{U} + \mathbf{E} \mathbf{U}]\)

- \(n \times \mathbf{E}_0 = 0, \text{ on } \partial \Omega,\)

Since \(f \in H^{-\frac{1}{2}}(\text{div}, \partial \Omega), \mathbf{U}\) must be in \(H(\text{curl}, \Omega)\) so that \(f \in (H^{-1}(\Omega))^3\).

Considering the duality pairing between \(H(\text{curl}, \Omega)\) and \((H^{-1}(\Omega))^3\), the equation for \(\mathbf{E}_0\) can be written as

\[a^{(p)}(\mathbf{E}_0, \Phi) = \langle F, \Phi \rangle.\]
So far we have

\[ E = U + E_0 , \quad U \in (H^1(\Omega))^3 ; n \times U = f , \quad f \in H^{-1/2}(\text{curl,} \Omega) \]

\[ a^{(P)}(E_0, \psi) = \langle F, \psi \rangle \quad \text{(\#)} \]

Now, seek \( E_0 \) in the form (Helmholtz decomposition)

\[ E_0 = \psi + \text{grad} \phi , \quad \phi \in H^1_0(\Omega) , \quad \psi \in H^1_0(\text{curl,} \Omega) \cap H^1(\text{div,} \Omega) \]

So (\#) becomes

\[ a^{(P)}(\psi, \psi) + a^{(P)}(\text{grad} \phi, \psi) = \langle F, \psi \rangle , \]

whereby

\[ a^{(P)}(\psi, \psi) + b^{(P)}(\phi, \psi) = \langle F, \psi \rangle , \quad \forall \psi \in H^1_0(\text{curl,} \Omega) \]

where

\[ b^{(P)} : H^1_0(\Omega) \times H^1_0(\text{curl,} \Omega) \rightarrow \mathbb{R} \]

is the bilinear form defined as

\[ b^{(P)}(\phi, \psi) := -w^2 \int_\Omega (\text{curl}(\beta \text{grad} \phi) + \varepsilon \text{grad} \phi) \cdot \psi \, dx \]

Eliminate the electrostatic potential \( \phi \).

Select the test function \( \psi \in H^1_0(\text{curl,} \Omega) \), so that

\[ \psi = \text{grad} \xi , \quad \xi \in H^1_0(\Omega) \]

(hence \( \text{curl} \psi = 0 \)), integrate by parts using the divergence theorem

\[ \int_\Omega \text{div} (\phi) \psi \, dx = \int_{\partial \Omega} \phi \cdot n(\psi) \, ds , \]

use also that \( \text{div} \text{curl} = 0 \), and after some algebra, get

\[ w^2 \left[ \int_\Omega \text{div}(\beta \text{curl} \phi + \varepsilon \phi) \cdot \xi \, dx + \int_{\partial \Omega} \varepsilon \text{grad} \phi \cdot \text{grad} \xi \, ds \right] = \int_{\partial \Omega} F \cdot \text{grad} \xi \, ds \]
If we choose \( \beta \) so that
\[
\text{div}(\beta \text{curl} \mathbf{e} + \mathbf{e} \mathbf{e}) = 0 \quad (\Leftrightarrow \text{div}(\mathbf{e} \mathbf{e}) = 0)
\]
or, its equivalent weak form
\[
\int_{\Omega} (\beta \text{curl} \mathbf{e} + \mathbf{e} \mathbf{e}) \cdot \text{grad} \mathbf{f} \, \text{d}x = 0, \quad \forall \mathbf{f} \in H^1_0(\Omega)
\]
the previous relation becomes
\[
w^2 \int_{\Omega} \mathbf{e} \cdot \text{grad} \mathbf{f} \cdot \text{grad} \mathbf{g} \, \text{d}x = \int_{\Omega} \mathbf{F} \cdot \text{grad} \mathbf{g} \, \text{d}x,
\]
which is the weak form of an elliptic equation for \( \mathbf{f} \).

Let us make a subtle remark here: although the embeddings of \( H(\text{curl}, \Omega) \) (as well as of \( H(\text{div}, \Omega) \)) and of their intersections with ker\text{curl} or ker\text{div} are not compact for "regular", bounded \( \Omega \), it is known that the space
\[
N^{(m)} = \{ \mathbf{u} \in H_0(\text{curl}, \Omega) \mid \text{div}(\mathbf{e} \mathbf{u}) = \text{grad}(\beta \mathbf{e}), \quad \text{curl} \mathbf{u} \in L^2(\Omega) \}
\]
is compact in \((L^2(\Omega))^3\) (Leis for smooth domains, Weck/Weber/R.Picard for more general)
under our assumptions on \( \mathbf{e}, \beta \).

Return to (4) and write it as
\[
\tilde{\alpha}(\phi, \xi) := w^2 \int_{\Omega} \mathbf{e} \cdot \text{grad} \phi \cdot \text{grad} \xi \, \text{d}x = \int_{\Omega} \mathbf{F} \cdot \text{grad} \xi \, \text{d}x
\]
where the form
\[
\tilde{\alpha} : H^1_0(\Omega) \times H^1_0(\Omega) \rightarrow \mathbb{C}
\]
is (clearly) bounded \& bilinear.

Then use Lax-Milgram to obtain that there exists a unique \( \phi \in H^1_0(\Omega) \) solving \( \tilde{\alpha}(\phi, \xi) = \int_{\Omega} \mathbf{F} \cdot \text{grad} \xi \, \text{d}x \), and that it satisfies
\[
\|
\text{grad} \phi
\|_{L^2(\Omega)} \leq C \|
\mathbf{F}
\|_{H(\text{div}, \Omega)} \|
\phi
\|_{H^1(\Omega)} \leq C \|
\mathbf{F}
\|_{H^{-1/2}(\text{div}, \Omega)} \|
\phi
\|_{H^{1/2}(\text{div}, \Omega)}
\]
Classical Lax-Milgram Lemma

**H**: Hilbert space

\[ a : H \times H \to \mathbb{C} \]

bounded (continuous) \[ |a(x,y)| \leq \gamma \|x\| \|y\| \]

coercive \[ a(x,x) \geq \delta \|x\|^2 \]

Sesquilinear form \[ a(x+y, z+w) = a(x,z) + a(x,w) + a(y,z) + a(y,w) \]

\[ a(px, qy) = p \overline{q} a(x, y) \]

Then, there exists \( A : H \to H \), \( A \in \mathcal{L}(H, H) \) defined by

\[ a(x, y) = \langle Ax, y \rangle \]

with \( |A| = \|A\| = \sup_{x, y \neq 0} \frac{|a(x, y)|}{\|x\| \|y\|} \leq \gamma \) \( \forall x, y \in H \).

Moreover,

\[ a(A^{-1}x, y) = \langle x, y \rangle, \]

with \( \|A^{-1}\| \leq \frac{1}{\delta} \) \( \forall x, y \in H \).

Then, the variational problem \( a(u, v) = \langle f, v \rangle \) has a solution in \( H \), for all \( f \in H' \).

\[
\begin{aligned}
-\Delta u &= f, \Omega \\
\frac{\partial u}{\partial n} &= 0, \partial \Omega
\end{aligned}
\]

**Weak Form**: \( \hat{a}(u, v) = \langle f, v \rangle, \forall v \in H_0^1(\Omega) \)

\[ \hat{a}(u, v) = \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx \]

\[ \langle f, v \rangle = \int_{\Omega} f(x)v(x) \, dx \]

Gårding's Inequality: \( \exists C \geq 0 \): \( \hat{a}(u, v) \geq C \|u\|_{H^1(\Omega)}^2 - C \|u\|_{L^2(\Omega)}^2 \)

for all \( u \in H_0^1(\Omega) \)

Poincaré's Inequality: \( \|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)} \)

\( C = C(\Omega) \)
5 Determination of $e$.

Substitute $\phi$ in

$$
curl(\theta \:\text{curl}(e + \text{grad}\phi)) - \omega^2 \left[ \text{curl}(\nabla e(e + \text{grad}\phi)) + \nabla e \cdot \text{curl}(e + \text{grad}\phi) + e(e + \text{grad}\phi) \right] =
$$

so, get

$$
curl(\theta \:\text{curl}\:U) - \omega^2 \left[ \text{curl}(\nabla e \:U) + \nabla e \cdot \text{curl}\:U + e \:U \right]
$$

F

$$
curl(\theta \:\text{curl}\:U) - \omega^2 \left[ \text{curl}(\nabla e) + \nabla e \cdot \text{curl}\:U + e \:U \right] =
$$

$$
= F + \omega^2 \left[ \text{curl}(\nabla e \:\text{grad}\phi) + e \:\text{grad}\phi \right]
$$

or, in variational notation

$$
a^{(B)}(e, \psi) = (F, \psi) + \omega^2 \left( \text{curl}(\nabla e \:\text{grad}\phi), \psi \right) + \omega^2 \left( e \:\text{grad}\phi, \psi \right), \quad \forall \psi \in H_0(\text{curl}, \Omega).
$$

6 Rewrite the above as

$$
a^{(B)}_{\theta}(e, \psi) = (\theta \:\text{curl}(e), \psi) + \omega^2 \left( \text{curl}(\nabla e \:\text{grad}\phi), \psi \right) + \omega^2 \left( e \:\text{grad}\phi, \psi \right)
$$

$$
- \omega^2 \left( \text{curl}(\nabla e), \psi \right) - 2\omega^2 \left( \nabla e \cdot \text{curl}\:U, \psi \right) - 2\omega^2 \left( e \:U, \psi \right) =
$$

$$
= (F, \psi) + \omega^2 \left( \text{curl}(\nabla e \:\text{grad}\phi), \psi \right) + \omega^2 \left( e \:\text{grad}\phi, \psi \right),
$$

and note that, by our assumptions on $\theta, e, \phi, \mu,$

$a^{(B)}(e, \psi)$ is a bounded, coercive, bilinear form on

$H(\text{curl}, \Omega) \times H_0(\text{curl}, \Omega)$.

Hence, by Lions-Milgram, for any $F \in (H_0^{-1}(\Omega))^3$,

the variational problem

$$
a^{(B)}_{\theta}(e, \psi) = (F, \psi), \quad \forall \psi \in H_0(\text{curl}, \Omega)
$$

has a unique solution.
Perturbation of coercive form - Fredholm Theory

Define the space \( N^{(p)} \) as in page 9, and recall that it is compactly embedded in \( (L^2(\Omega))^3 \).

Let \( \phi \) be the solution of the variational problem

\[
\alpha_{+}^{(p)}(\phi, \psi) = -w^{2} \left[ (\text{curl}(\varepsilon \text{curl} \phi), \psi) + 2(\varepsilon \text{curl} \phi, \psi) + 2(\varepsilon \psi, \psi) \right],
\]

for all \( \psi \in N^{(p)} \).

Define the operator \( K : (L^2(\Omega))^3 \to (L^2(\Omega))^3 : K\varepsilon = \phi \).

i) \( K \) is well-defined (by Lax-Milgram)

ii) \( \varepsilon \in N^{(p)} \Rightarrow K\varepsilon \in N^{(p)} \)

\[
\text{div} \text{RHS} = 0, \text{ since } \varepsilon \in N^{(p)} \Rightarrow \text{div} (\varepsilon \text{curl} K\varepsilon + 2\varepsilon) = 0
\]

and \( \text{div} \text{curl} = 0 \)

\[
\text{div} \text{LHS} = \text{div} (\varepsilon K\varepsilon + \varepsilon \text{curl} K\varepsilon), \text{ since } \text{div} \text{curl} = 0
\]

Hence \( \text{div}(\varepsilon K\varepsilon + \varepsilon \text{curl} K\varepsilon) = 0 \Rightarrow K\varepsilon \in N^{(p)} \)

(This formal argument becomes rigorous, since - by the regularity of the solution of the above variational problem - \( \varepsilon \in H_{\text{curl}}(\Omegaa) \)).

iii) \( K \) maps bounded subsets of \( (L^2(\Omega))^3 \) into bounded sets of \( N^{(p)} \).

iv) \( N^{(p)} \subset (L^2(\Omega))^3 \)

So, essentially, \( K \) is compact.
Now define the map \( G \) as: given \( \phi \), let
\[ G\phi \] be the solution of the variational problem
\[ a_+^{(p)}(G\phi, \psi) = (F, \psi) + \omega^2 \left[ \left( \text{curl}(\text{Re grad} \phi), \psi \right) + \left( \text{Re grad} \phi, \psi \right) \right], \]
for all \( \psi \in N^{(p)} \).

is well-defined (by Lax-Milgram).

In view of the relations defining \( K \) and \( G \), the relation
\[ a_+^{(p)}(e, \psi) - \omega^2 \left[ \left( \text{curl}(\text{Re e}), \psi \right) - 2 \left( \text{Re curl} e, \psi \right) - 2 (e, \psi) \right] = \]
\[ (F, \psi) + \omega^2 \left[ \left( \text{curl}(\text{Re grad} \phi), \psi \right) + \left( \text{Re grad} \phi, \psi \right) \right] \]
of page 11, can be written as
\[ a_+^{(p)}(e, \psi) + a_+^{(p)}(Ke, \psi) = a_+^{(p)}(G\phi, \psi) \]
for all \( \psi \in H_0(\text{curl}, \Omega) \).

Since \( a_+^{(p)} \) is bilinear, the above relation can be written as
\[ (I + K)e = G\phi. \]

Since \( K \) is a compact operator, the Fredholm Alternative ascertains that for every \( G\phi \), the inhomogeneous equation \((I + K)e = G\phi\) has a unique solution depending continuously on \( G\phi \), if the homogeneous equation
\[ (I + K)e = 0 \]
has only the trivial solution, which is the same as
\[ a_+^{(p)}(e + Ke, \psi) = (0, \psi). \]

QED (cf. page 11)
Remark

A not very different approach for the proof of the solvability result uses a mixed variational formulation and the Babuška-Brezzi generalization of the Lax-Milgram Lemma (next page).

A suitable variational formulation is needed:

$E$ is decomposed as

$$ E = \nabla + e + \text{grad} \phi $$

$\nabla \in (H^1_0(\Omega))^3 : n \times \nabla = f$

$e \in H^1_0(\text{curl}, \Omega) \cap H(\text{div}, \Omega)$

$\phi \in H^1_0(\Omega)$

$$ a^{(n)}(e, w) + b^{(n)}(\phi, w) = a^{(n)}(U, w), \quad \forall w \in H^1_0(\text{curl}, \Omega) $$

$$ b^{(n)}(\psi, e) = 0, \quad \forall \psi \in H^1_0(\Omega) $$

$$ |a^{(n)}(e, e)| \geq c_1 \|\text{curl} e\|^2_{(L^2(\Omega))^3} - c_2 \|e\|^2_{(L^2(\Omega))^3}, \quad c_1 > 0 $$

$$ |b^{(n)}(\psi, \text{grad} \phi)| \geq c_3 \|\psi\|^2_{H^1_0(\Omega)}, \quad c_3 > 0 $$

The Babuška-Brezzi inf-sup condition is satisfied for $b^{(n)}$.

Further,

$$ \ker(b^{(n)}) = N^{(n)} \subseteq (L^2(\Omega))^3, \quad \text{so} \quad a^{(n)} \text{ is a compact perturbation of a coercive form on } \ker(b^{(n)}) $$

The Fredholm Alternative completes the proof.
Generalization of Lax–Milgram Lemma

\( H_1, H_2 \): Hilbert spaces

\( a: H_1 \times H_2 \to \mathbb{C} \): bounded, sesquilinear form such that

(i) \( \exists \, \alpha > 0 : \inf \sup_{u \in A_1, v \in A_2} |a(u, v)| \geq \alpha > 0 \)

where

\[ A_1 := \{ u \in H_1 : \| u \|_{H_1} = 1 \} \]
\[ A_2 := \{ v \in H_2 : \| v \|_{H_2} = 1 \} \]

\( \) \{ INF-SUP or BABUŠKA-BREZZI CONDITION \}

(ii) \( \forall \, 0 \neq v \in H_2 : \sup_{u \in H_1} |a(u, v)| > 0 \)

Then for every \( f \in H_2' \), there exists a unique \( u \in H_1 \):

\[ a(u, v) = (f, v), \forall v \in H_2 \]

The solution to this problem satisfies the bound

\[ \| u \|_{H_1} \leq \frac{C}{\alpha} \| f \|_{H_2'} \]

We also have the following result (mixed variational problem)

Let \( H \) and \( S \) be Hilbert spaces, and let \( a: H \times H \to \mathbb{C} \) and \( b: H \times S \to \mathbb{C} \) be bounded, sesquilinear forms such that \( \exists \alpha > 0 : (T-coercivity of a(.\cdot)) \)

\[ |a(u, u)| \geq \alpha \| u \|^2_H \] for all \( u \in T := \{ u \in H : b(u, s) = 0, \forall s \in S \} \), (i)

where \( \alpha \) is independent of \( u \), and \( \exists \, \beta > 0 : \)

\[ \sup_{u \in H} \frac{b(u, p)}{\| u \|_H} \geq \beta \| p \|_S \], (Babuška-Brezzi condition) (ii)

where \( \beta \) is independent of \( p \).

Let \( f \in H' \) and \( g \in S' \). Then, the problem of finding \( u \in H \) and \( p \in S \)

such that

\[ a(u, \Phi) + b(\Phi, p) = f(\Phi), \forall \Phi \in H \]
\[ b(u, s) = g(s), \forall s \in S \]

has a unique solution, and

\[ \| u \|_{H} + \| p \|_{S} \leq C(\| f \|_{H'} + \| g \|_{S'}) \]
The Eigenvalue Problem

\[ L^{(p)} : H_0(\text{curl}, \Omega) \to (H^{-1}(\Omega))^3 \]

\[ \nabla \mapsto L^{(p)} \nabla := \text{curl}(\text{curl} \nabla) - \omega^2 \left[ \text{curl}(\text{curl} \nabla \omega) + \beta \text{curl} + \epsilon \nabla \right] \]

\[ \alpha^{(p)}(E, \psi) = \langle L^{(p)} E, \psi \rangle, \forall \psi \in H_0(\text{curl}, \Omega) \]

In terms of the above, the eigenvalue problem is to find \( \omega \) such that the problem

\[ L^{(p)} E = 0, \]

or

\[ \alpha^{(p)}(E, \psi; \omega^2) = 0, \forall \psi \in H_0(\text{curl}, \Omega), \]

admits nontrivial solutions.

Assume that \( \beta > 0 \) is a constant and, further, that it is small (physically relevant assumption).

- First consider the case \( \beta = 0 \).

\[ L^{(0)} \nabla := \text{curl} \left( \mu^{-1} \text{curl} \nabla \right) - \omega^2 \nabla \epsilon \nabla \]

\[ E = \epsilon + \text{grad} \phi \quad \left((L^2(\Omega))^3 = \text{grad} H_0^1(\Omega) + H(\text{div}0, \Omega)\right) \]

\[ \epsilon \in N^{(0)} = \{ u \in H_0(\text{curl}, \Omega) : \text{div}(\epsilon u) \in L^2(\Omega) \} \]

\[ \phi \in H_0^1(\Omega) \]

\[ \alpha^{(0)}(E, \psi; \omega^2) = 0, \forall \psi \in N^{(0)} \]

**Friedrich's Inequality:** \( \Omega : \text{bounded, simply connected, Lipschitz} \)

Then \( \exists C > 0 : \forall \omega \in \Omega \), we have

\[ \| W \|_{(L^2(\Omega))^3} \leq C \left[ \| \text{curl} W \|_{L^2(\Omega)}^3 + \| \nabla \times W \|_{L^2(\Sigma)}^3 \right] \]

Note that in our case \( \Sigma = \emptyset \).
If \( \omega^2 = 0 \) we have that
\[
(\text{curl}(\mu^{-1}\text{curl}E), \psi) = 0
\]
\[
\Rightarrow \ (\mu^{-1}\text{curl}E, \text{curl}\psi) = 0 \quad \forall \ \psi \in N^{(0)}
\]
\[
\Rightarrow \ (\mu^{-1}\text{curl}E, \text{curl}\psi) = 0 \quad \forall \ + \in N^{(0)}
\]
\[
e \in N^{(0)} \quad \Rightarrow \quad e = 0.
\]

So \( \omega^2 = 0 \) is an eigenvalue of infinite multiplicity, with corresponding eigenfunctions \( E = \text{grad}\phi, \phi \in H_0^1(\Omega) \).

These eigenvalues are not considered to be physically relevant since no sources are present \( \text{div}(EE) \) must be 0. Then \( \text{div}(E\text{grad}\phi) = 0 \) and since \( \phi \in H_0^1(\Omega) \Rightarrow \phi = 0 \).

Rewrite the eigenvalue problem (\( \omega^2 > 0 \) now) as
\[
\alpha^{(0)}_+ (\varepsilon, \psi; 1) = -b^{(0)} (\varepsilon, \psi; \omega^2 + 1), \quad \forall \ \psi \in N^{(0)}
\]
and define the operator \( K : (L^2(\Omega))^3 \to (L^2(\Omega))^3 \) which acts on a function \( w \), giving the solution of
\[
\alpha^{(0)}_+ (Kw, \psi; 1) = -b^{(0)} (w, \psi; 1), \quad \forall \ \psi \in N^{(0)}.
\]

Similarly as for \( K \), it follows that \( K \) is compact.

Further, using the equivalent, weighted by \( \varepsilon \), norm of \( (L^2(\Omega))^3 \), \( K \) becomes self-adjoint.

The eigenvalue problem can be rewritten as
\[
K e = \frac{1}{1 + \omega^2} e,
\]
and then, by the Hilbert-Schmidt Theory, we get that

- There is an infinite, discrete set of eigenvalues \( \omega_j^2, j \in \mathbb{N} \), such that \( \lim_{j \to \infty} \omega_j^2 = \infty \), \( 0 < \omega_1^2 \leq \omega_2^2 \leq \ldots \).
Consider now the case $B \neq 0$.

**Lemma:** \( \exists \, w > 0 \) and \( B_c > 0 \) : \( \forall \, B \in [0, B_c] \),

0 is not an eigenvalue of \( L^{(B)} \) in \( \Omega \)
(with the PEC boundary condition on \( \partial \Omega \)).

Indeed, since \( \theta - \mu^{-1} = -\omega^2 \varepsilon B^2 \), we have

\[
((L^{(B)} - L^{(0)})v, v) = -\mu^2 \omega^2 \int_\Omega |\text{curl} v|^2 dx +
\]

\[
-\mu \omega^2 \int_\Omega \varepsilon \text{curl} v \cdot \nu dx - \mu \omega^2 \int_\Omega \varepsilon V \cdot \text{curl} \overline{V} dx
\]

\[
\left( \int_\Omega \text{curl} v \cdot W dx - \int_\Omega V \cdot \text{curl} W dx = \int_{\partial \Omega} (n \times V) ((n \times W) \times n) ds \right)
\]

whereby,

\[
|((L^{(B)} - L^{(0)})v, v)| \leq C \|v\|_{H_0^1(\Omega, \mathbb{R})}
\]

for an appropriate constant \( C \).

Hence \( \lim_{B \to 0} \|L^{(B)} - L^{(0)}\|_{L_2(\partial \Omega)} = 0 \).

The result follows, based on the properties of the problem for \( B = 0 \).
Main References